Equivalence Relations and Operations

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This lecture finishes the topic of equivalence relations (section 8.5 of Rosen) and shows how to define operations on equivalence classes.

1 Announcements

Reminder: quiz Wednesday.

2 Proving that a relation is an equivalence relation

So far, I've assumed that we can just look at a relation and decide if it's an equivalence relation. Suppose someone asks you (e.g. on an exam) to prove that something is an equivalence relation. These proofs just use techniques you've seen before. Let's do a couple examples.

First, remember that we had defined the set of fractions F to contain all fractions $\frac{x}{y}$ where $x, y \in \mathbb{Z}$ and $y \neq 0$. We constructed the rational numbers from the fractions by defining certain pairs of fractions to be equivalent. Specifically, our equivalence relation $\sim$ was defined by: $\frac{x}{y} \sim \frac{p}{q}$ if and only if $xq = yp$.

Let's show that $\sim$ is an equivalence relation.

Proof: Reflexive: For any x and y, $xy = xy$. So the definition of $\sim$ implies that $\frac{x}{y} \sim \frac{x}{y}$.

Symmetric: if $\frac{x}{y} \sim \frac{p}{q}$ then $xq = yp$, so $yp = xq$, so $py = qx$, which implies that $\frac{p}{q} \sim \frac{x}{y}$.

Transitive: Suppose that $\frac{x}{y} \sim \frac{p}{q}$ and $\frac{p}{q} \sim \frac{s}{t}$. By the definition of $\sim$, $xq = yp$ and $pt = qs$. So $xqt = ypt$ and $pty = qsy$. Since $ypt = pty$, this means that $xqt = qsy$. Cancelling out the q's, we get $xt = sy$. By the definition of $\sim$, this means that $\frac{x}{y} \sim \frac{s}{t}$.

Since $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation.

Notice that the proof has three sub-parts, each showing one of the key properties. Each part involves using the definition of the relation, plus a small amount of basic math. The reflexive case is very short. The symmetric case is often short as well. Most of the work is in the transitive case.

There are relations for which one or the other property is hard to prove. Sometimes even reflexive can require some work. But it's more typical for these proofs to be almost mechanical.

3 Proving antisymmetry

Here's another example of a relation, this time an order (not an equivalence) relation. Consider the set of intervals on the real line $J = \{(a, b) \mid a, b \in \mathbb{R} \text{ and } a \leq b\}$. Define the containment relation C as follows:

$$(a, b)\ C\ (c, d) \text{ if and only if } a \leq c \text{ and } d \leq b$$

Let's show that C is antisymmetric.

Notice that C is a relation on intervals, i.e. pairs of numbers, not single numbers. If we substitute the definition of C into the definition of antisymmetric, what we need to show is that $(a, b)\ C\ (c, d)$ and $(c, d)\ C\ (a, b)$ implies that $(a, b) = (c, d)$.

So, suppose that we have two intervals (a, b) and (c, d) such that $(a, b)\ C\ (c, d)$ and $(c, d)\ C\ (a, b)$. By the definition of C, $(a, b)\ C\ (c, d)$ implies that $a \leq c$ and $d \leq b$. Similarly, $(c, d)\ C\ (a, b)$ implies that $c \leq a$ and $b \leq d$.

Since $a \leq c$ and $c \leq a$, $a = c$. Since $d \leq b$ and $b \leq d$, $b = d$. So $(a, b) = (c, d)$.

4 Rays in space

How about a more complex example? Let's model the set of rays from the origin out to infinity in 3D. Constructions similar to this are often used to model physical systems such as cameras.

Let $A = \mathbb{R}^3 - \{(0, 0, 0)\}$. That is, A is 3D space, except for the origin. We would like to treat two points as equivalent if they lie on the same ray. That is:

> $(x, y, z) \sim (p, q, r)$ if and only if there is a positive real number λ such that $(x, y, z) = \lambda(p, q, r)$ i.e. $x = \lambda p$, $y = \lambda q$, and $z = \lambda r$.

Each equivalence class of $\sim$ is a ray from the origin (except for the origin itself). For example, $[(3, 2, 7)]$ is the ray passing through the point $(3, 2, 7)$ and it contains other points like $(1.5, 1, 3.5)$ and $(3.75, 2.5, 8.75)$.

Let's prove that $\sim$ is an equivalence relation. [In lecture, I'll probably only actually do transitivity.]

Reflexive: If $\lambda = 1$, then $(x, y, z) = \lambda(x, y, z)$. So $(x, y, z) \sim (x, y, z)$.

Symmetric: Suppose $(x, y, z) \sim (p, q, r)$ Then there is a positive real λ such that $(x, y, z) = \lambda(p, q, r)$. But then $(p, q, r) = \frac{1}{\lambda}(x, y, z)$. So $(p, q, r) \sim (x, y, z)$. If λ is a positive real, so is $\frac{1}{\lambda}$.

Transitive: Suppose that $(x, y, z) \sim (p, q, r)$ and $(p, q, r) \sim (a, b, c)$. Then there are positive reals λ and κ such that $(x, y, z) = \lambda(p, q, r)$ and $(p, q, r) = \kappa(a, b, c)$. Substituting the second equation into the first, we get that $(x, y, z) = \lambda\kappa(a, b, c)$. Since λ and κ are positive reals, so is $\lambda\kappa$. So by the definition of $\sim$, $(x, y, z) \sim (a, b, c)$.

Since $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation.

5 Proving that an operation is well-defined

Getting back to our rational number example, let's look at how to define operations on this new type of numbers. Given that we know how to do arithmetic on integers, we can define addition of fractions by

$$\frac{x}{y} + \frac{p}{q} = \frac{xq + py}{yq}$$

We'd like to apply this definition to our new rational numbers, by simply throwing equivalence class brackets around the inputs and outputs:

$$[\frac{x}{y}] + [\frac{p}{q}] = [\frac{xq + py}{yq}]$$

To work right, the output of this definition should not depend on which representative we pick for the input equivalence classes. E.g. $[\frac{2}{3}] + [\frac{10}{2}]$ should be equal to (say) $[\frac{-4}{-6}] + [\frac{5}{1}]$. In math jargon, we say that we want to make sure the operation is *well-defined* on the equivalence classes.

Suppose we defined a function $\oslash$ by $[\frac{x}{y}] \oslash [\frac{p}{q}] = [\frac{xy}{pq}]$. Then $[\frac{2}{3}] \oslash [\frac{10}{2}] = [\frac{6}{20}]$. Suppose we pick different representatives for the two input equivalence classes: $[\frac{-4}{-6}] \oslash [\frac{5}{1}] = [\frac{24}{5}]$. This produces a different output equivalence class because $\frac{6}{20} \not\sim \frac{24}{5}$. So this function isn't well-defined, a diplomatic way of saying that this is a buggy definition that doesn't work right.

Let's prove that our definition of rational addition is well-defined. Suppose that we pick two different representatives for each input: $\frac{x}{y} \sim \frac{v}{w}$ and $\frac{p}{q} \sim \frac{r}{s}$. Then we need to show that the two outputs we get are equivalent: $\frac{xq+py}{yq} \sim \frac{vs+wr}{ws}$

Since $\frac{x}{y} \sim \frac{v}{w}$, $xw = yv$. So $xwqs = yvqs$.

Since $\frac{p}{q} \sim \frac{r}{s}$, $ps = qr$. So $psyw = qryw$.

Adding these equations together: $xwqs + psyw = yvqs + qryw$. So $(xq + py)ws = (vs + wr)yq$. So, by the definition of $\sim$, $\frac{xq+py}{yq} \sim \frac{vs+wr}{ws}$.

6 Defining real numbers via equivalence classes

Now, suppose that we know about the integers (and maybe the rational numbers) and we would like to define the real numbers. One way to do this is using equivalence classes. To make this simple, let's just worry about defining the real numbers between 0 and 1. If you can do that, it's not hard to define the full set of real numbers.

Suppose $D = \{0, 1, 2, \ldots, 9\}$ is the set of decimal digits. Let's make a set S containing all infinite sequences of digits. Each sequence will represent the decimal representation of a real. Formally, each sequence is a function from the natural numbers to the digits. That is, $S = \{f \text{ a function } \mid f : \mathbb{N} \to D\}$

So S is the set of real numbers, right? Well, sort of. The difficulty is that many reals have two representations, one ending in an infinite sequence of zeros and one ending in an infinite sequence of nines. So, we need to set up an equivalence relation $\sim$ that relates each sequence ending in an infinite string of nines with the sequence that contains the next larger number followed by an infinite sequence of zeros. $0.8789999\ldots \sim 0.879000\ldots$. The real numbers between zero and one are then the equivalence classes of this relation $\sim$, i.e. we merge each pair of equivalent decimals into a single object.

7 Mathematical glue

We can also use equivalence classes to cut-and-paste geometrical objects. For example, consider the interval $I = [0, 1]$ of the real line. You can think of it like a piece of string. If we were to glue together the two ends of the string, we would get a little loop or circle of string.

To do this mathematically, we create an equivalence relation S in which most points are in their own one-element equivalence class, but 0 and 1 are in the same equivalence class. That is, that xSy if $x = y$, or if $x = 0$ and $y = 1$, or $x = 1$ and $y = 0$. Then I/S is the equivalence classes of I under the relation S and I/S is a loop.

Now, let's see how to construct a Möbius strip. A physical Möbius strip is made by taking a strip of paper, giving it a half twist, and taping the ends together. This object is neat, because it has only one side (not two) and only one edge.

To define a Möbius strip mathematically, we select a rectangular piece of the real plane, e.g. $[0,1] \times [0,1]$, i.e. all the points with both coordinates between 0 and 1 (inclusive). Let's call this patch of 2D surface I^2.

We can define various equivalence relations on I, thereby glueing I to itself in different ways. First, suppose we define R by $(x,y)R(p,q)$ if

- $(x,y) = (p,q)$ or
- $y = q$, $x = 0$, and $p = 1$, or
- $y = q$, $x = 1$, and $p = 0$.

Then the set of equivalence classes I^2/R is mostly like I^2, except that we've glued together the left and right edges of I^2. So it's a tube.

Similarly, if we also glue the top of I^2 to the bottom, we'll get a torus. If we merge all the edge points of I^2 into one point, we'll get a sphere.

Now, suppose that we set up the following equivalence relation T:

- $(x,y) = (p,q)$ or
- $y = 1 - q$, $x = 0$, and $p = 1$, or
- $y = 1 - q$, $x = 1$, and $p = 0$.

This merges points on the left and right edges, but attaches the one edge upside down. Or, viewed another way, we put a half twist into the strip of surface before we glue the ends. When we merge equivalence points to create I^2/T, we have a Möbius strip.

If you carry on like this, you can build up all sorts of interesting and strange deformed surfaces, starting with pieces of the ordinary 2D plane.

Filling in the details of examples like these could be a bit tricky, beyond what we plan to put on homeworks or exams. But they illustrate why the idea of equivalence relations is useful in later mathematics and CS courses.