CS 173: Discrete Structures

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Adjacent Vertices (Neighbors)

Two vertices, *u* and *v* in an undirected graph *G* are called adjacent (or neighbors) in *G*, if {*u*, *v*} is an edge of *G*.

An edge *e* connecting *u* and *v* is called incident with vertices *u* and *v*, or is said to connect *u* and *v*.

The vertices *u* and *v* are called endpoints of edge {*u*, *v*}.

Directed Adjacency

- Let *G* be a directed (possibly multi-) graph
- Let *e* be an edge of *G* that is (or maps to) (*u*,*v*).

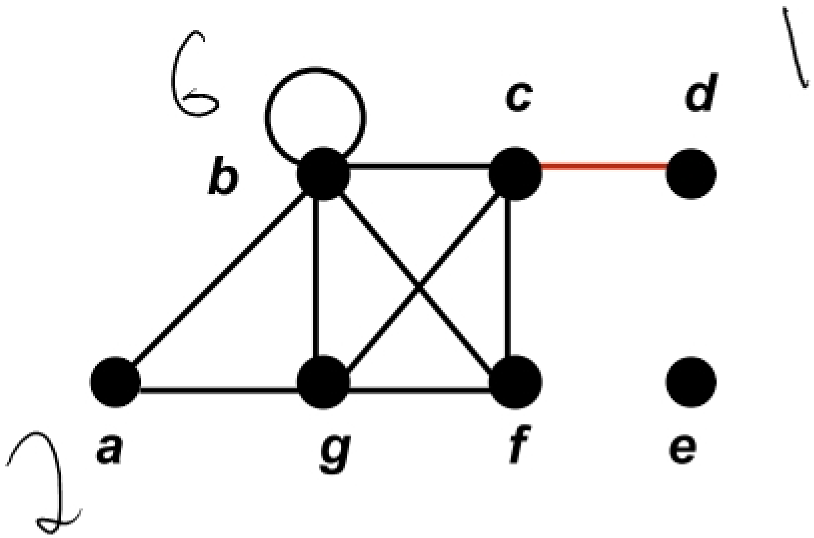
Then we say:

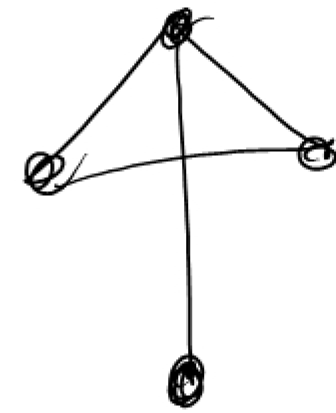
- *u* is *adjacent to v*, *v* is *adjacent from u*
- *e comes from* u, e *goes to* v.
- *e connects* u *to v*, *e goes from u to v*
- the *initial vertex* of *e* is *u*
- the *terminal vertex* of *e* is *v*

Degree of a vertex

The degree of a vertex in an undirected graph is

- the number of edges incident with it, except that
- a loop at a vertex contributes twice to the degree of that vertex





Directed Degree

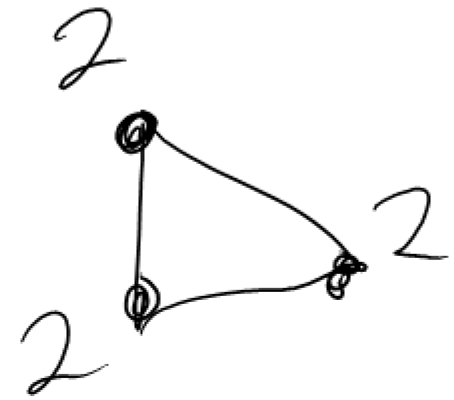
- Let G be a directed graph, v a vertex of G.
  - The in-degree of v, $deg^{-}(v)$, is the number of edges going to v.
  - The out-degree of v, $deg^{+}(v)$, is the number of edges coming from v.
  - The degree of v, $deg(v) = deg^{-}(v)+deg^{+}(v)$, is the sum of v's in-degree and out-degree.

Handshaking Theorem

3edges

Let G = (V, E) be an undirected graph G with e edges. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$



$\Sigma deg = 6$

"The sum of the degrees over all the vertices equals

____"

NOTE: This applies even if multiple edges and loops are present.

Handshaking Theorem

- **Corollary:** Any undirected graph has an

even # of

vertices of odd degree.

Special Graph Structures

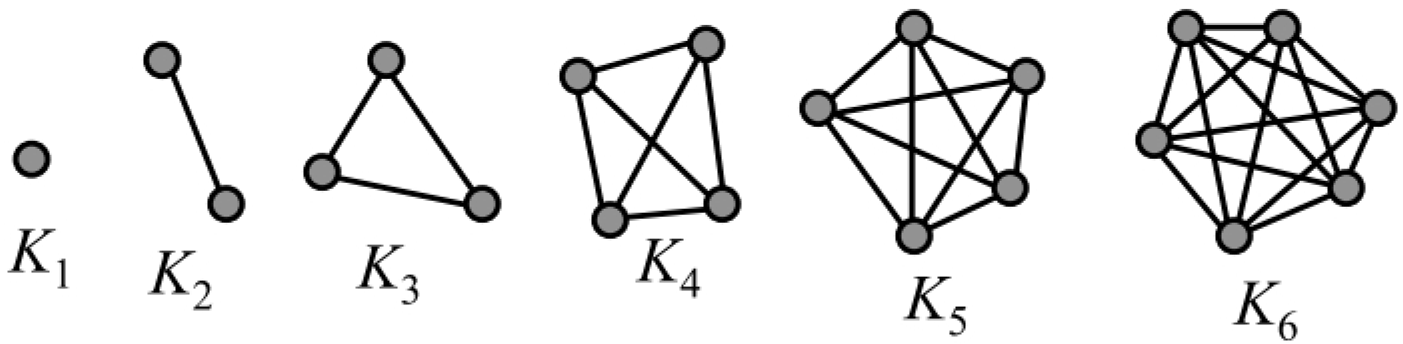
Special cases of undirected graph structures:

- Complete graphs K_n
- Cycles C_n
- Wheels W_n
- n-Cubes Q_n
- Bipartite graphs
- Complete bipartite graphs $K_{m,n}$

Complete Graphs

K_n

- For any $n \in \mathbf{N}$, a *complete graph* on *n* vertices, K_n, is a simple graph with *n* nodes in which every node is adjacent to every other node: $\forall u,v \in V\colon u \neq v \leftrightarrow \{u,v\} \in E$.



$$\binom{n}{2} = \frac{n!}{(n-2)!\,2}$$

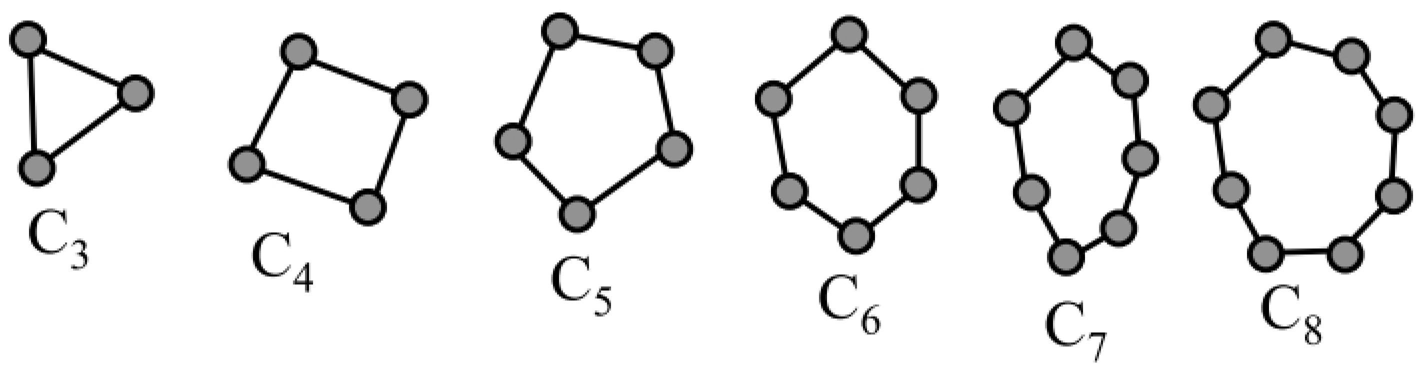
$$\boxed{= \frac{n(n-1)}{2}}$$

- How many edges does K_n have?

Cycles

C_n

- For any $n \geq 3$, a *cycle* on n vertices, C_n, is a simple graph where $V=\{v_1,v_2,\ldots,v_n\}$ and $E=\{\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{n-1},v_n\},\{v_n,v_1\}\}$.

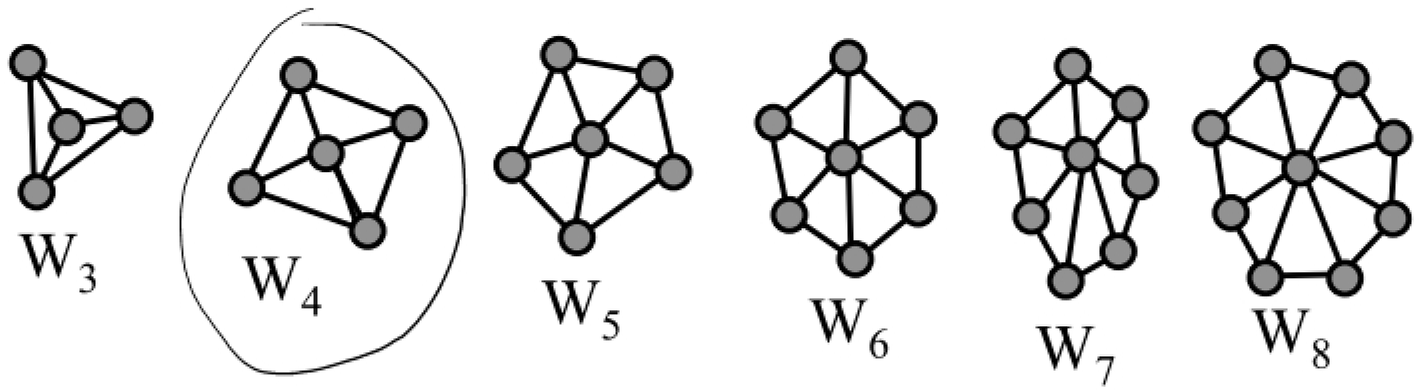


How many edges are there in C_n? $\boxed{n}$

Wheels

W_n has $n+1$ vertices

- For any $n \geq 3$, a *wheel* W_n, is a simple graph obtained by taking the cycle C_n and adding one extra vertex v_{hub} and n extra edges $\{\{v_{hub}, v_1\}, \{v_{hub}, v_2\}, \ldots, \{v_{hub}, v_n\}\}$.



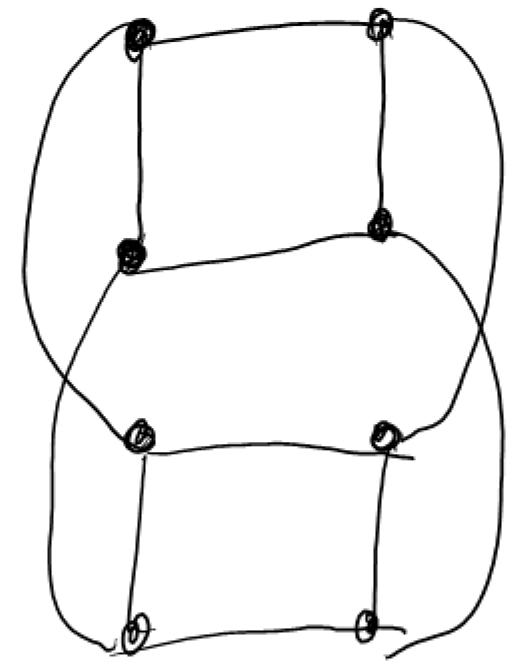
How many edges are there in W_n?

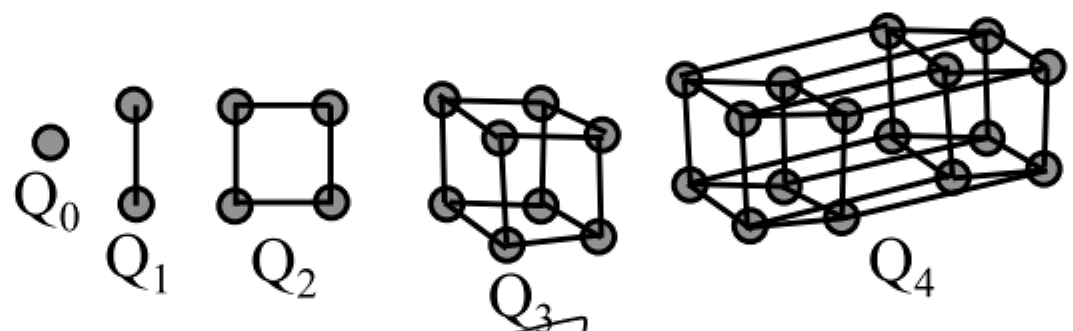
$\boxed{2n}$

n-cubes (hypercubes)

$Q_n = Q_{n-1} + Q_{n-1}$

- For any $n \in \mathbf{N}$, the hypercube Q_n is a simple graph consisting of two copies of Q_{n-1} connected together at corresponding nodes. Q_0 has 1 node.





edges in 2 Q_{n-1}'s

Number of vertices: $\boxed{2^n}$. *Number of edges: Exercise to try!*

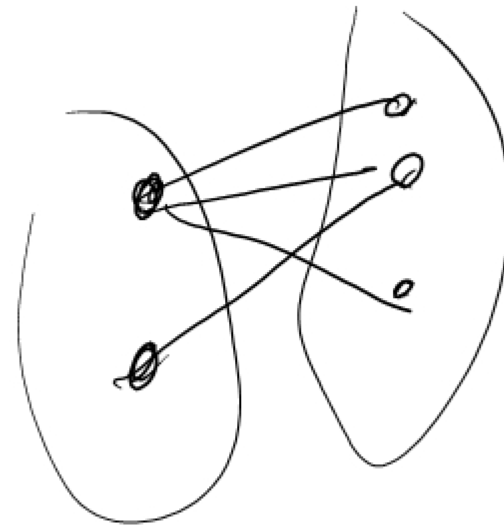
$$E(n) = 2E(n-1) + 2^{n-1}$$

$$E(1) = 1$$

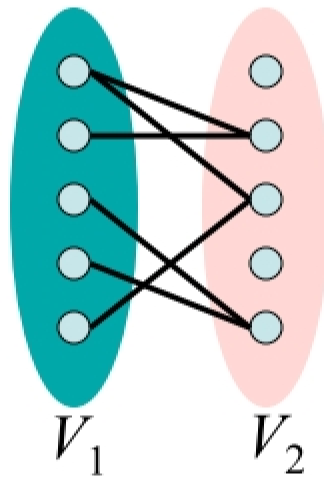
Bipartite Graphs

- **Def'n.:** A graph $G=(V,E)$ is *bipartite* (two-part) iff
  - $V = V_1 \cup V_2$
  - $V_1 \cap V_2 = \emptyset$

  $\forall e \in E$: $\exists v_1 \in V_1, v_2 \in V_2$: $e = \{v_1, v_2\}$.



- The graph canbe divided into two parts in such a way that all edges go between the two parts.



Complete Bipartite Graphs

$K_{m,n}$

- For $m,n \in \mathbf{N}$, the *complete bipartite graph* $K_{m,n}$ is a bipartite graph where

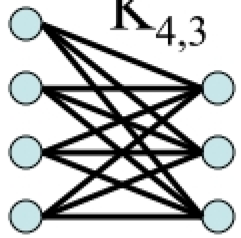
  $|V_1| = m$

  $|V_2| = n$

  and $E = \{\{v_1,v_2\} \mid v_1 \in V_1 \wedge v_2 \in V_2\}$.

$K_{m,n}$ has $\underline{m+n}$ nodes and $\underline{mn}$ edges.

$K_{4,3}$

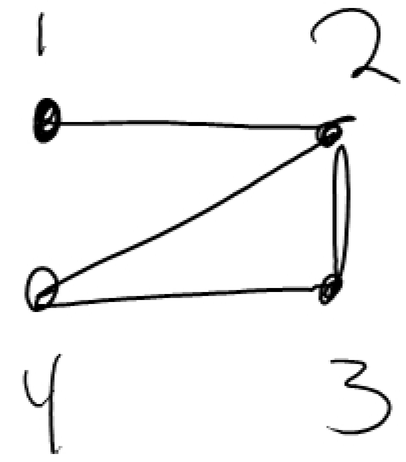


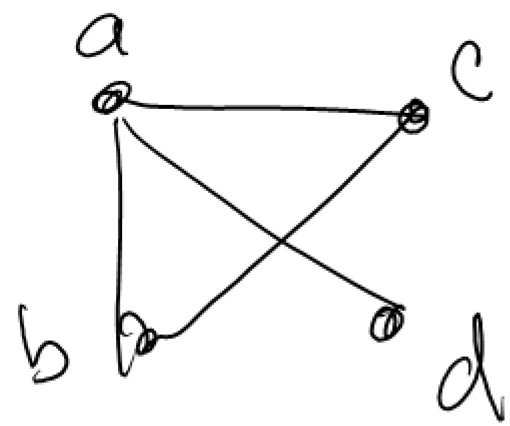
$|V_1| = m$ $\quad$ $|V_2| = n$

Graph Isomorphism

- Simple graphs $G_1=(V_1, E_1)$ and $G_2=(V_2, E_2)$ are *isomorphic* iff $\exists$ a bijection $f:V_1 \rightarrow V_2$ such that $\forall$ $a,b \in V_1$, a and b are adjacent in G_1 iff $f(a)$ and $f(b)$ are adjacent in G_2.

- f is the "renaming" function between the two node sets that makes the two graphs identical.





$1 \rightarrow d$

$2 \rightarrow a$

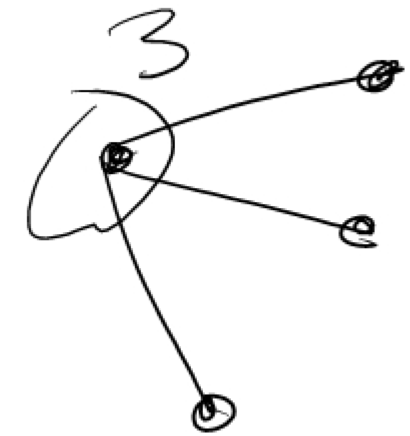
$3 \rightarrow b$

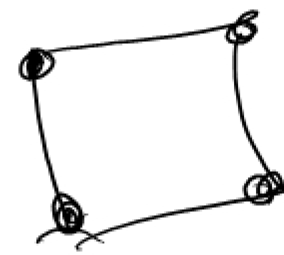
$4 \rightarrow c$

Graph Invariants under Isomorphism

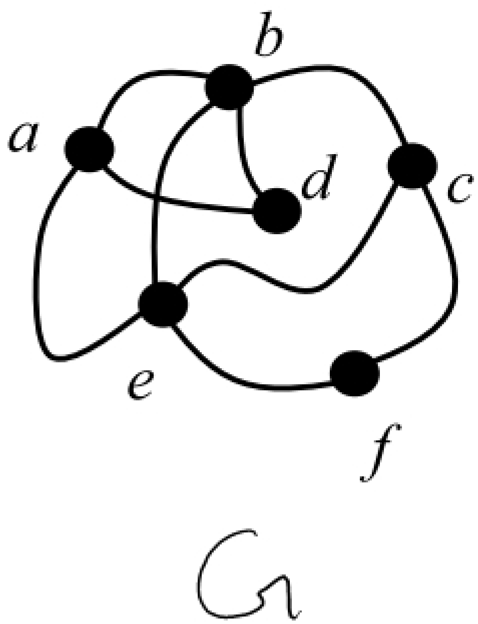
Necessary but not *sufficient* conditions for $G_1=(V_1, E_1)$ to be isomorphic to $G_2=(V_2, E_2)$:

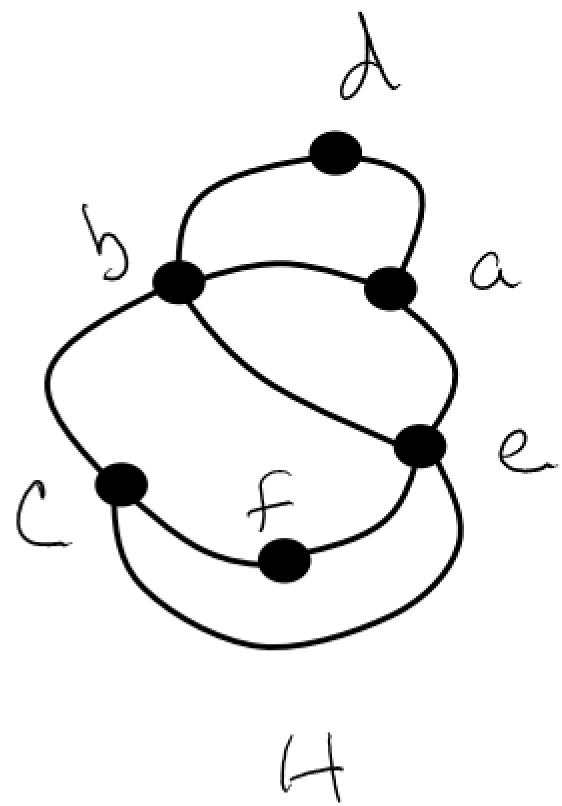
- We must have that $|V1|=|V2|$, and $|E1|=|E2|$.
- The number of vertices with degree *n* is the same in both graphs.
- For every proper subgraph *g* of one graph, there is a proper subgraph of the other graph that is isomorphic to *g*.



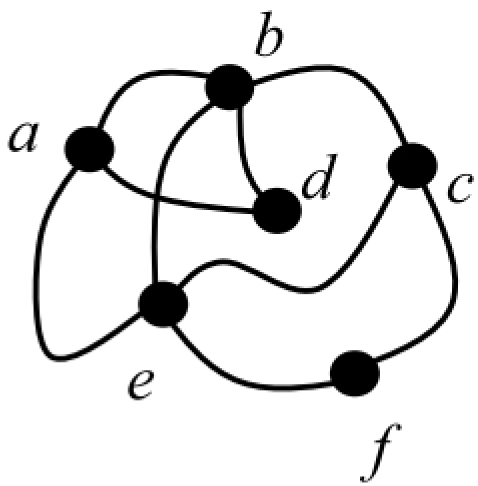


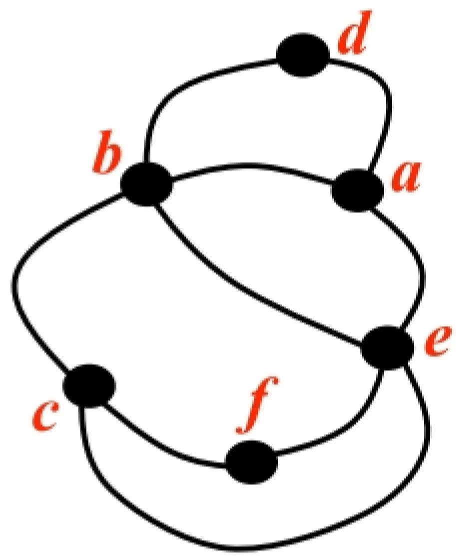
Isomorphism?





Isomorphism Example



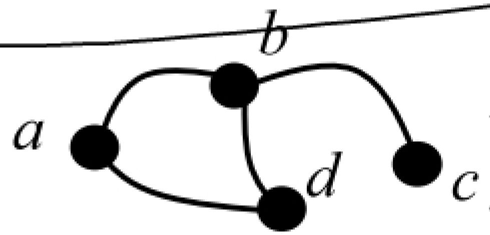


Graph Representations

symmetric

Adjacency Matrix

$\frac{n(n-1)}{2}$ entries



	a	b	c	d
a	0	1	0	1
b	1	0	1	1
c	0	1	0	0
d	1	1	0	0

Adjacency List

$a \to b, d$

$b \to a, c, d$

$c \to b$

$d \to a, b$

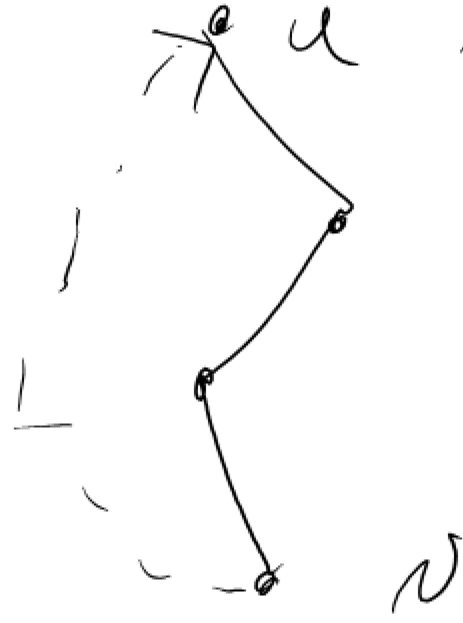
$2|E|$

Which is more space efficient?

Connectivity

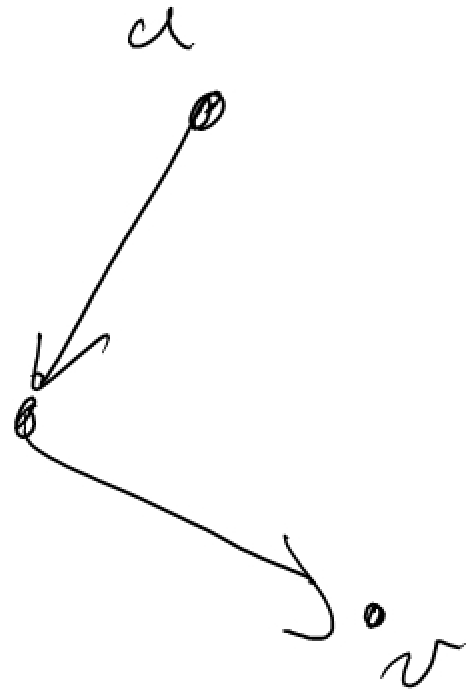
- In an undirected graph, a ***path of length n*** *from u to v is* sequence of adjacent edges from vertex *u* to vertex *v*.
- A path is a *circuit* if *u=v*.
- A path *traverses* the vertices along it.
- A path is *simple* if it contains no edge more than once.

path of length 3



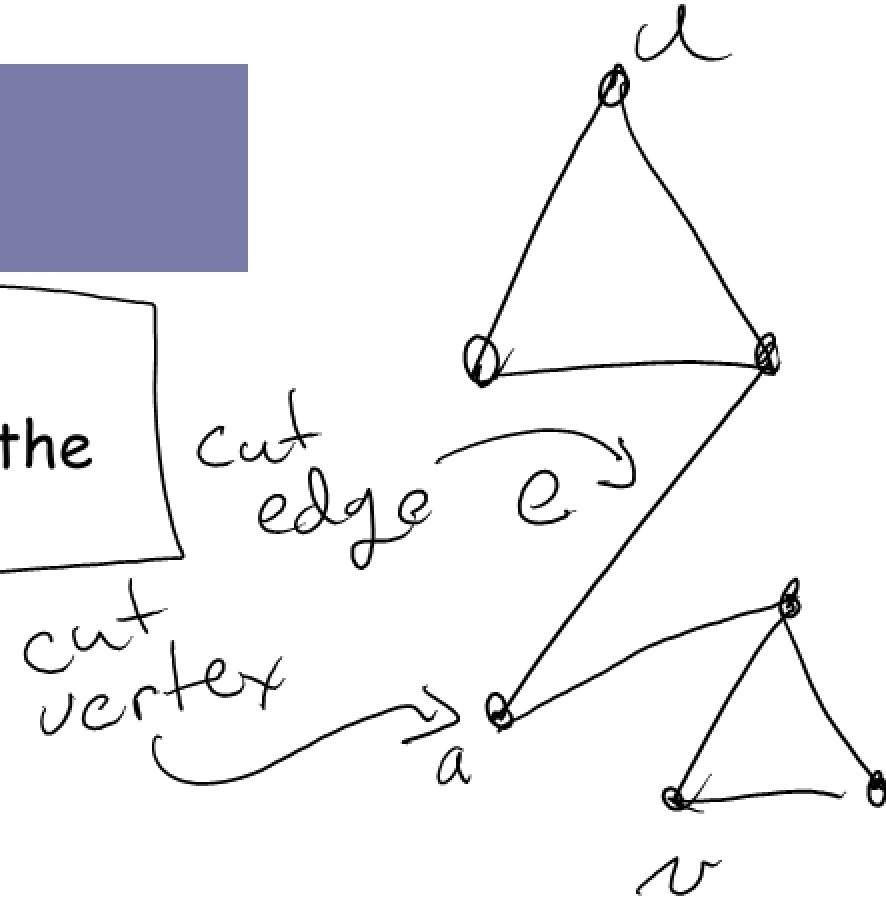
Paths in Directed Graphs

- Same as in undirected graphs, but the path must go in the direction of the arrows.



Connectedness

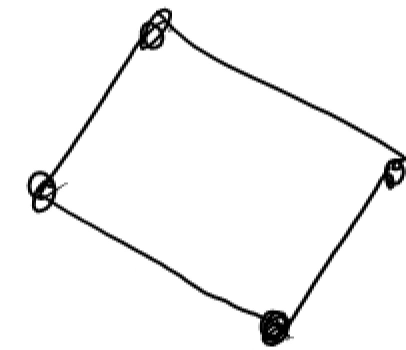
- An undirected graph is *connected* iff there is a path between every pair of distinct vertices in the graph.
- There is a *simple* path between any pair of vertices in a connected undirected graph.
- *Connected component*: connected subgraph
- A *cut vertex* or *cut edge* separates 1 connected component into 2 if removed.



Connectedness

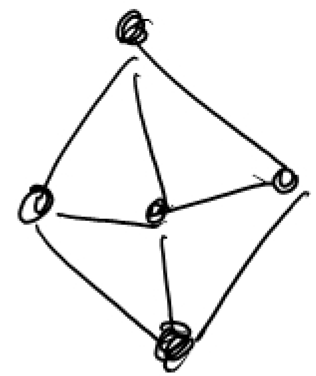
- How many edges can you remove before C_n disconnects?

  1 at most



- What about W_n?

  n at most

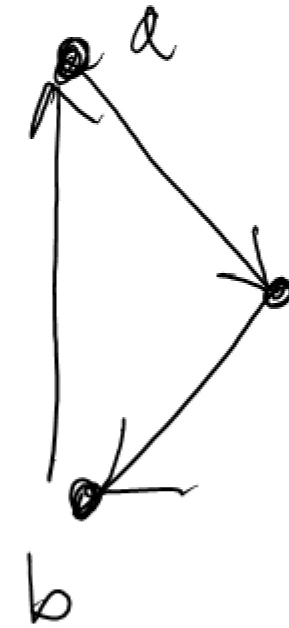


Directed Connectedness

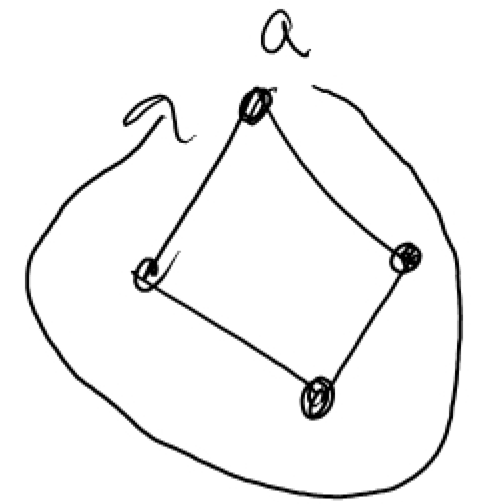
- A directed graph is *strongly connected* iff there is a directed path from *a* to *b* for any two verts *a* and *b*.

- It is *weakly connected* iff the underlying *undirected* graph (*i.e.*, with edge directions removed) is connected.

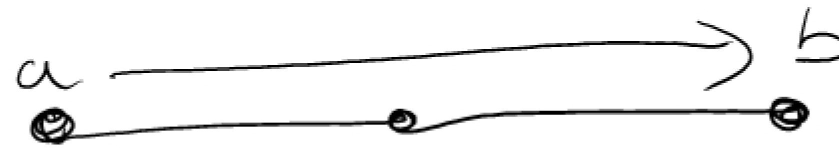
- Note *strongly* implies *weakly* but not vice-versa.



Euler & Hamilton Paths



- An ***Euler circuit*** in a graph G
  is a simple circuit containing every edge of G.

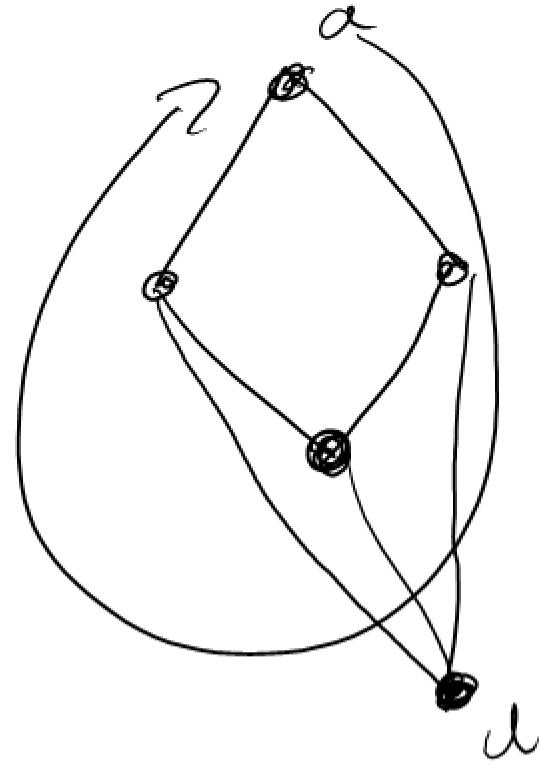


- An ***Euler path*** in G
  is a simple path containing every edge of G.

- A ***Hamilton circuit***
  is a circuit that traverses each vertex in G exactly once.

- A ***Hamilton path***
  is a path that traverses each vertex in G exactly once.

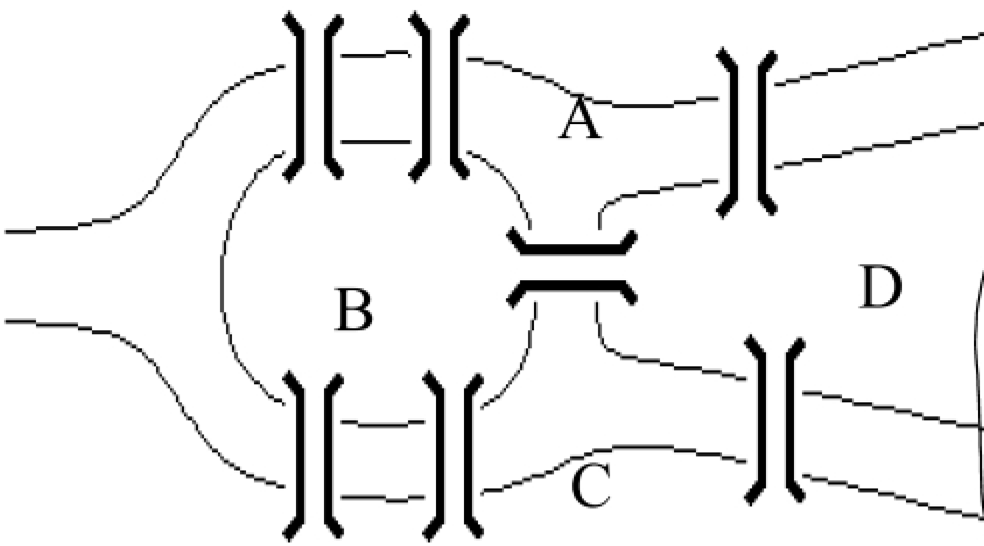
Euler Circuit Theorems



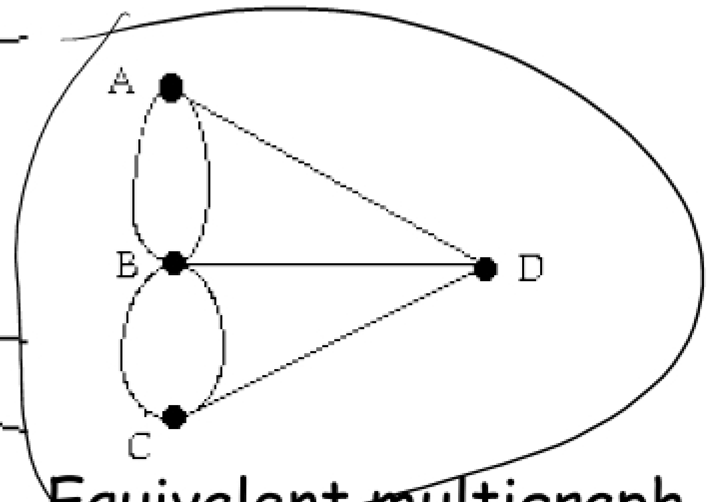
- **Theorem:** A connected multigraph has an Euler circuit iff each vertex has even degree.
  - **Proof:**
    - (→)The circuit contributes 2 to degree of each vertex
    - (←) By construction using algorithm on page 636

Bridges of Königsberg Problem

- Can we walk through town, crossing each bridge exactly once, and return to start?



The original problem



Equivalent multigraph

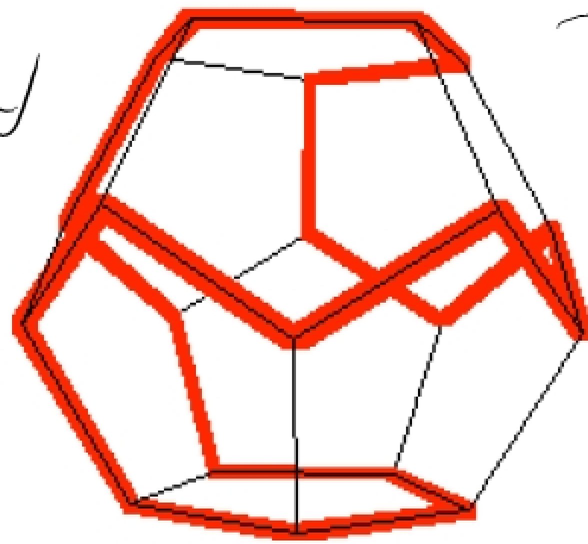
Euler Path Theorem

Theorem: A connected multigraph has an Euler path iff it has exactly 2 vertices of odd degree.

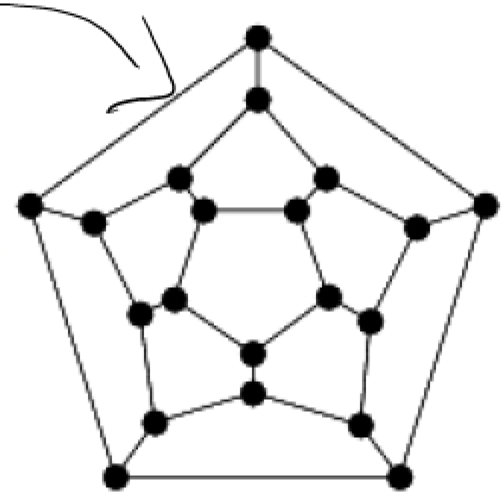
- One is the start, the other is the end.
- Does not have an Euler circuit.

Round-the-World Puzzle

- Can we traverse all the vertices of a dodecahedron, visiting each once?
- Puzzle invented by William Rowan Hamilton (1857)



Dodecahedron puzzle



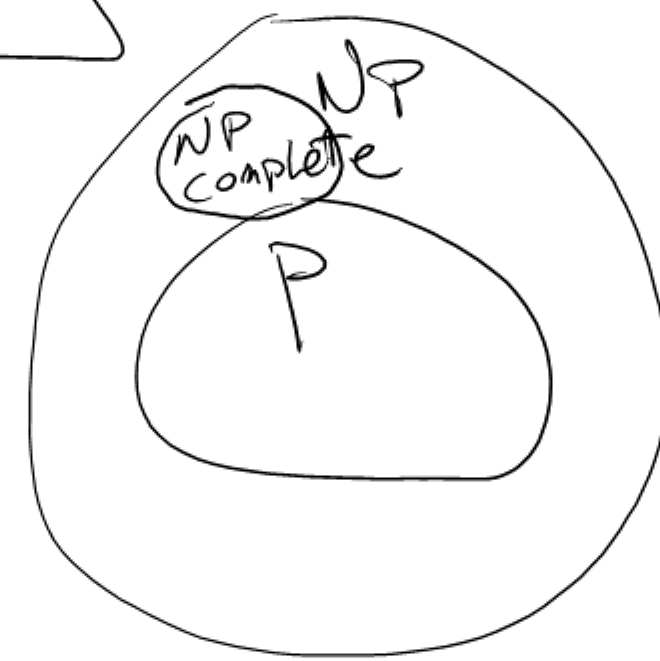
Equivalent graph

Hamiltonian Circuit Theorems

- **Dirac's theorem:** If (but not only if) G is connected, simple, has $n \geq 3$ vertices, and $\forall v\ \deg(v) \geq n/2$, then G has a Hamilton circuit.
- **Ore's corollary:** If G is connected, simple, has $n \geq 3$ nodes, and $\deg(u)+\deg(v) \geq n$ for every pair u,v of non-adjacent nodes, then G has a Hamilton circuit.
- You won't be tested on these
- What to remember:
  - What a Hamiltonian circuit is
  - Some easily computed sufficient conditions are known
  - No efficient algorithm for detecting Ham circuit in general

NP-complete

- Given a simple graph *G*, does *G* have a Hamiltonian circuit?
- Best known algorithms have exponential complexity $\boxed{O(C^{|V|})}$
- This problem has been proven to be NP-Complete
- If an algorithm for solving it in polynomial time were found
  - *all* problems in class NP solvable polynomial time
  - P=NP=Turing Award
  - …but probably no such algorithm exists



$\boxed{NP \neq P\ ?}$

Graph Theory in pop culture

- Psychologist Stanley Milgram (1967) studied social networks
  - examined the avg path length of acquaintances in US
  - tested via letter delivery to strangers via friends
    - half arrived in less than 6 edge traversals

- "Six degrees" application on facebook
- Six is probably inaccurate (but close).
- How does the graph really look?
- Important ramifications for epidemiology
- Plus, there's a good chance everyone in this room is descended from ancient royalty
  - So we have that going for us, which is nice