CS 173: Discrete Structures

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Conditional Probability Revisited

S = Sample Space

$E \subseteq S$

$F \subseteq S$

Let E and F be events with Pr(F) > 0. The conditional probability of E given F, denoted by Pr(E|F) is defined to be:

$$\Pr(E|F) = \Pr(E \cap F)/\Pr(F).$$

What if $E \cap F = \varnothing$? $\Pr(E|F) = 0$

When would F not effect the Pr(E)?

when does $\Pr(E|F) = \Pr(E)$?

if $F = S$, $\Pr(E|F) = \Pr(E)$

Independence

The events E and F are *independent* if and only if $Pr(E \cap F) = Pr(E) \times Pr(F)$.

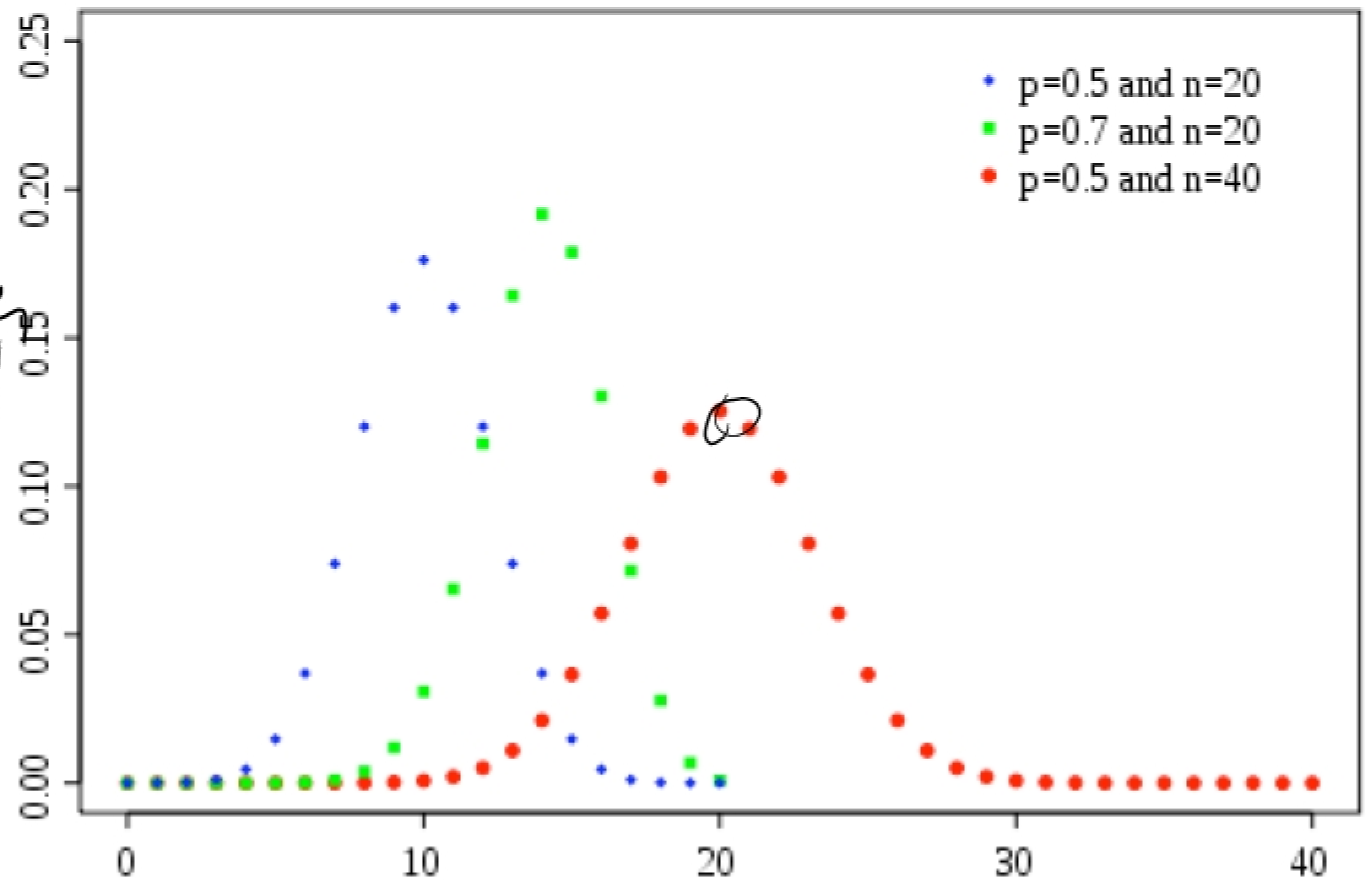
If two events are independent, what is $Pr(E|F)$?

Conditional Pr

$$P(E|F) = \frac{Pr(E \cap F)}{Pr(F)} = \frac{Pr(E)\,Pr(F)}{Pr(F)}$$

Binomial Distribution

$$b(k;n,p) = C(n,k)p^k(1-p)^{n-k}$$



prob. of k successes

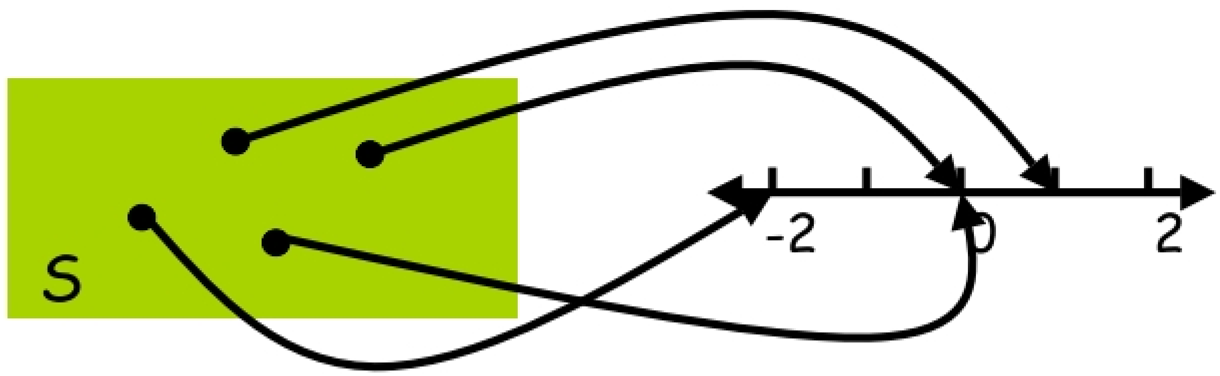
k

prob. of k successes in n trials

pr(success)=p

Random Variables

For a given sample space S, a *random variable* is any real valued function on S.



Suppose our experiment is a roll of 2 dice. S is set of pairs.

- X = sum of two dice. X((2,3)) = 5
- Y = difference between two dice. Y((2,3)) = 1
- Z = max of two dice. Z((2,3)) = 3

Random Variables

Example: Do you ever play the game Racko?

Suppose we are playing a game with cards labeled 1 to 20, and we draw 3 cards. We bet that the maximum card has value 17 or greater. What's the probability we win the bet?

Let r.v. X denote the maximum card value. The possible values for X are 3, 4, 5, …, 20.

i	3	4	5	6	7	8	9	…	20
Pr(X = i)	?	?	?	?	?	?	?		?

Filling in this box would be a pain. We look for a general formula.

Random Variables

X is value of the highest card among the 3 selected. 20 cards are labeled 1 through 20.

We want Pr(X = i), i = 3,…20.

Denominator first: How many ways are there to select the 3 cards? C(20,3)

How many choices are there that result in a max card whose value is i? C(i-1,2)

Pr(X = i) = C(i-1, 2) / C(20,3) These are the table values.

We win the bet if the max card, X is 17 or greater. What's the probability we win? Pr(X = 17) + Pr(X = 18) + Pr(X = 19) + Pr(X = 20) ≈0.51

Expected Value

Let X be a discrete r.v. with set of possible values D. The *expected value of X is:*

$$E(X) = \sum_{x \in D} x \cdot \Pr(X = x)$$

This is the weighted average value of the function X

The probability of X=x is used as the weight for each x

Imagine I have an unbalanced die with the probabilities

X=1,2,3,4 occur with probability .2

X= 5,6 occur with probability .1

Rolling a fair die

$X = i$

$$E(x) = \sum_{i=1}^{6} \left(\frac{1}{6}\right) i = \frac{1}{6}\left(\frac{6 \cdot 7}{2}\right) = 3.5$$

$$.2(1+2+3+4) + .1(5+6) = 3.1$$

Expected Value

trials

prob of success

Let X be a binomial r.v. with parameters n and p.

That is, X is the number of "successes" on n trials where each trial has probability of success p.

X = # successes in an outcome

What is E(X)?

First we need Pr(X = k) = C(n, k) $p^k (1-p)^{n-k}$

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

Expected Value

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \frac{k \cdot n!}{(n-k)!k!} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!(k-1)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n} \frac{n-1!}{(n-k)!(k-1)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n} \binom{n-1}{k} p^{k-1} (1-p)^{n-k}$$

ignore

$$= np[p + (1-p)]^{n-1} = np$$

Expected Value

$f(x) + g(x) + \dots$

$X_1 + X_2 + \dots$

Let X_i, $i = 1,2,\dots,n$, be a sequence of random variables, and suppose we are interested in their sum. The sum is a random variable itself with expectation given by:

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

The proof of this is inductive and algebraic. You can find it in your book on page 382.

Expected Value

Suppose you all (n) put your cell phones in a pile in the middle of the room, and I return them randomly. What is the expected number of students who receive their own phone back?

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

Define for i = 1, … n, a random variable:

$$X_i = \begin{cases} 1 & \text{if student } i \text{ gets the right phone,} \\ 0 & \text{otherwise.} \end{cases}$$

k	0	1
$Pr(X_i=k)$	1-(1/n)	1/n

$E[X_i]$ =

a) 1/n
b) 1/2
c) 1
d) No clue

$E[X_i] = Pr(X_i = 1)$

Expected Value

Suppose you all (n) put your cell phones in a pile in the middle of the room, and I return them randomly. What is the expected number of students who receive their own phone back?

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

Define for i = 1, … n, a random variable:

$$X_i = \begin{cases} 1 & \text{if student } i \text{ gets the right phone,} \\ 0 & \text{otherwise.} \end{cases}$$

Define r.v. $X = X_1 + X_2 + \ldots + X_n$, and we want E[X].

$E[X] = E[X_1 + X_2 + \ldots + X_n]$

$= E[X_1] + E[X_2] + \ldots + E[X_n]$ $= 1/n + 1/n + \ldots + 1/n$ $= 1$

Bucket Sort

What is the lower bound on the time to sort N items using comparisons?

$\Omega(N \lg N)$

Distribution-based sorting can be faster

- It uses knowledge about the data to sort faster
- We assume we have N numbers in the range [0,r)
- Assume the N numbers are uniformly distributed in[0,r)

Bucket Sort

Bucket Sort(a1,…,aN)

- Divide the range [0,R) into N uniform subranges
- Place each ai into the subrange (bucket)
- Sort each bucket using insertion sort
- Concatenate the buckets together in order

Bucket Sort

n=10

10 buckets

Example: we have 10 integer numbers in [0,100)

4,3,15,22,28,78,99,56,81,29

Our buckets:

3 4	15	29 28 22			56		78	81	99
[0,9]	[10,19]	[20,29]	[30,39]	[40,49]	[50,59]	[60,69]	[70,79]	[80,89]	[90,99]

3,4 15 22,28 29 56 78 81 99

Bucket Sort Expected Performance

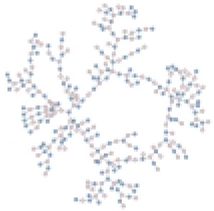
Let X_i be a random variable

For a given distribution of numbers into bucket
X_i= number in bucket i

Probability of k items in bucket i follows the binomial dist.

Math magic:
$E[X_i^2] = Var[X_i] + E[X_i]^2$
$E[X_i^2] = 1 - 1/n + 1^2 = O(1)$

With N buckets the expected time is O(n)



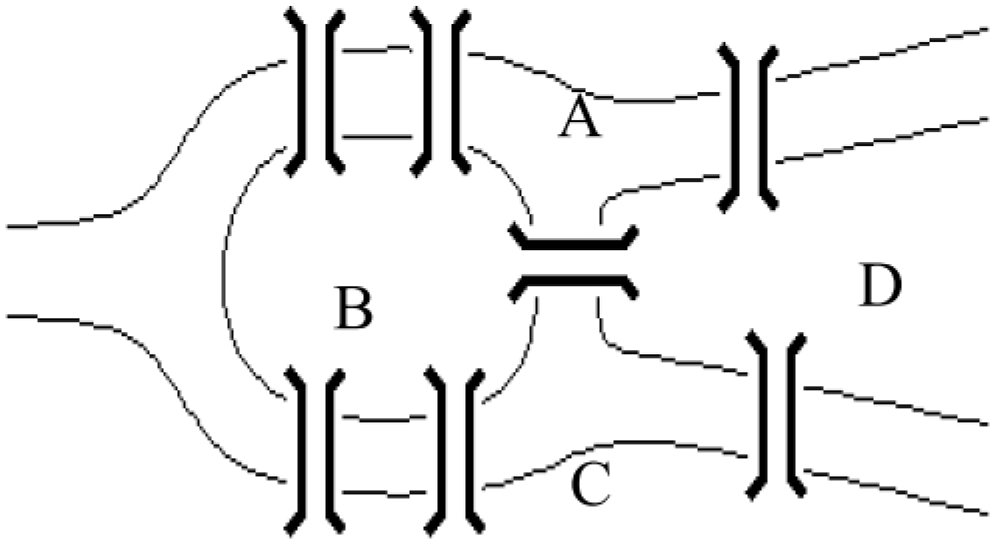
Graphs

- We'll try to get through sections 9.1 - 9.4 of the book
- Last major topic of the course
- Mathematics is not (only) the study of numbers
    - "the science that draws necessary conclusions"
    - Science of patterns
- We'll define a discrete structure called a graph
    - And then we'll prove things about
- But first…why?

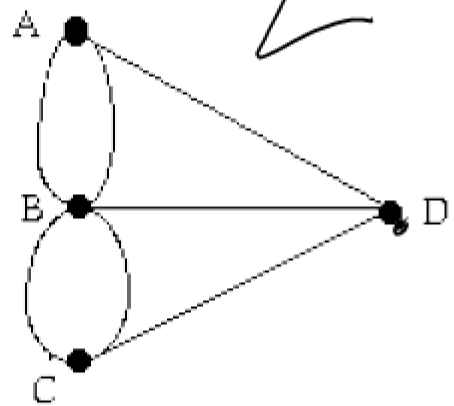
Bridges of Königsberg Problem

- Now Kaliningrad, Russia, divided into 4 parts by a river
- Can we walk through town, crossing each bridge exactly once, and return to start?
- Solved in 1736 by Leonhard Euler

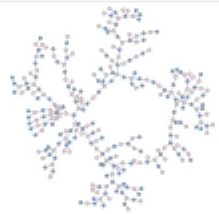
vertices = places
edges = bridges



The original problem



Equivalent graph



Other Applications of Graphs

- They can make people billionaires
  - PageRank
- Optimizing register assignment in program compilation
- Optimizing transportation logistics
- Modeling computer networks…or any network
- Used in bioinformatics (phylogenetic trees, proteomics)

Simple Graph

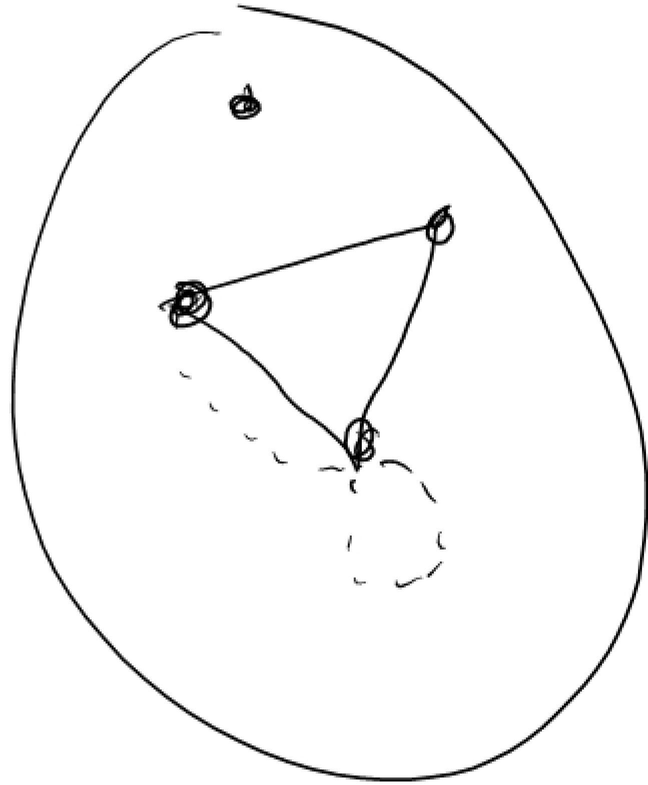
A simple graph $G = (V, E)$ consists of V, a nonempty set of vertices, and E, a set of unordered pairs of distinct elements of V called edges.

In this class:

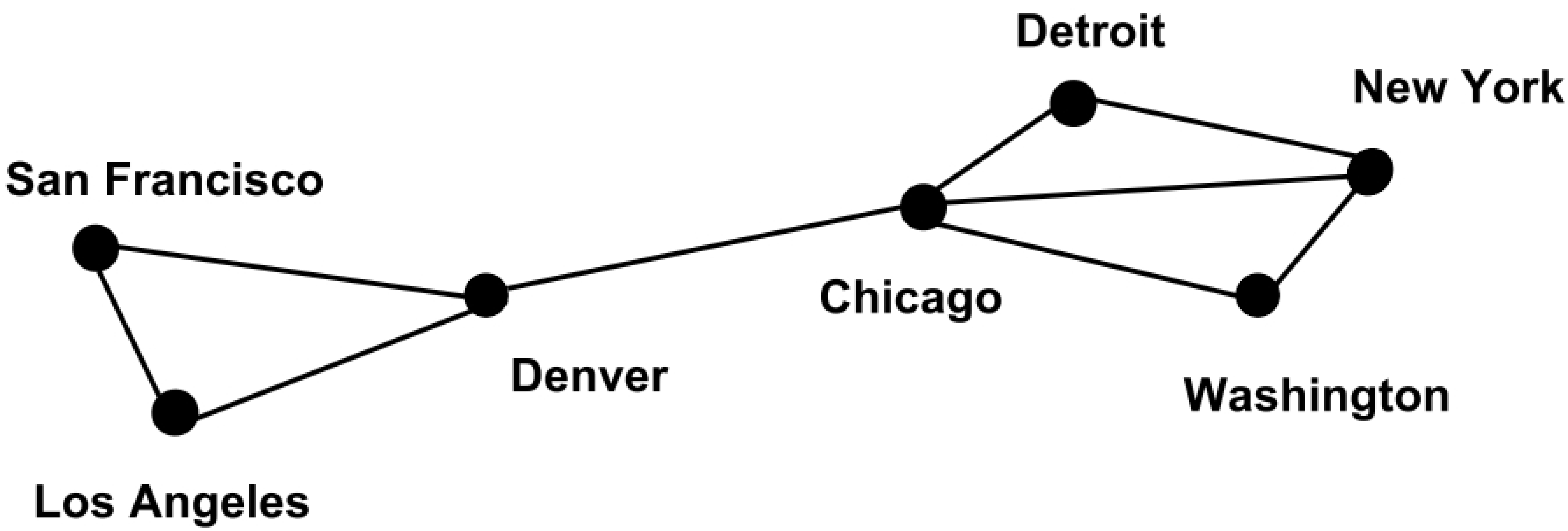
- When we say "graph" you should assume we mean a simple graph
- If we aren't talking about a simple graph, we will tell you that explicitly

In a multigraph $G = (V, E)$ two or more edges may connect the same pair of vertices.

In a pseudograph $G = (V, E)$ two or more edges may connect the same pair of vertices, and in addition, an edge may connect a vertex to itself.



A simple graph



A simple graph

SET OF VERTICES

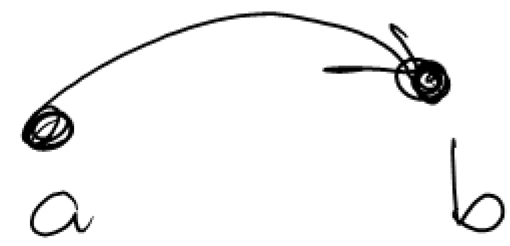
V = { Chicago, Denver, Detroit, Los Angeles,
New York, San Francisco, Washington }

SET OF EDGES

E = { {San Francisco, Los Angeles}, {San Francisco, Denver},
{Los Angeles, Denver}, {Denver, Chicago},
{Chicago, Detroit}, {Detroit, New York},
{New York, Washington}, {Chicago, Washington},
{Chicago, New York} }

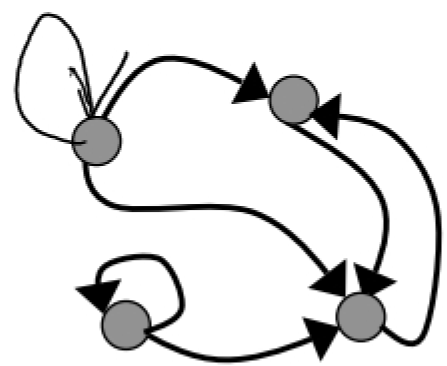
Directed Graphs

- Edges have a direction
- If you imagine walking on the graph
  - you can only move along edges in the given direction



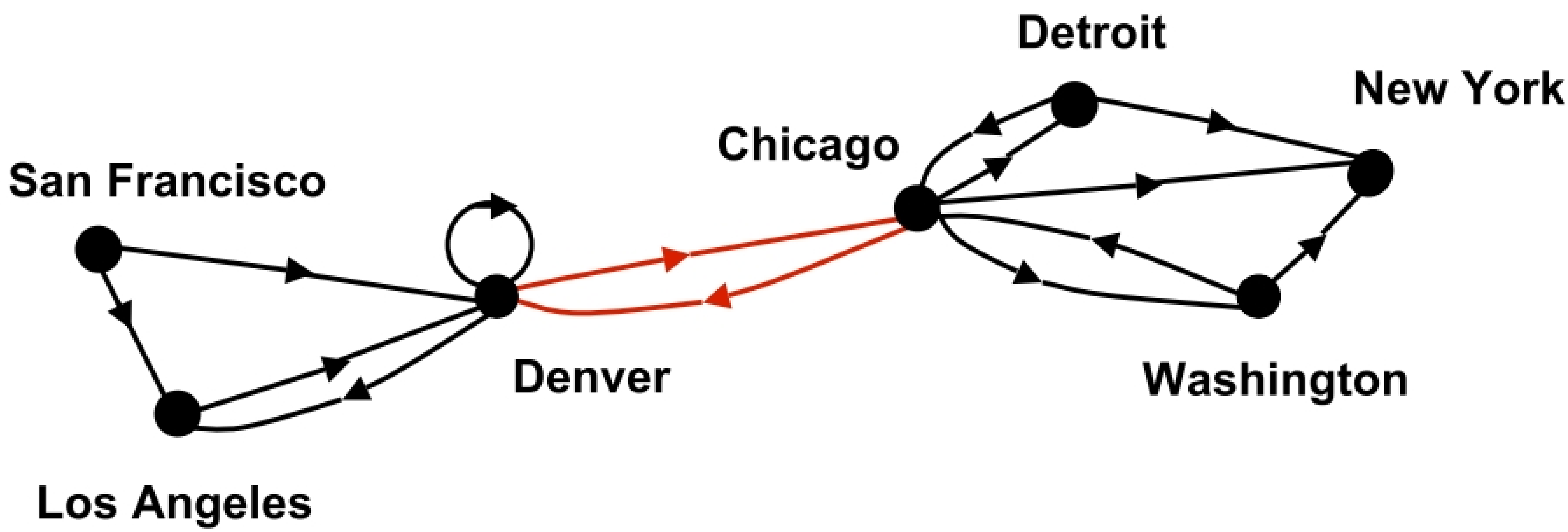
go from a to b but not b to a

- *A directed graph* (V,E) consists of
  - a set of vertices V
  - and a set of edges $E = \{(v_i,v_j),\ldots\}$
    where the edge goes from v_i ---> v_j
- *Notice that* $(v_i,v_j) \neq (v_j,v_i)$
- Self-loops are allowed



- *Di Graph example*: V = set of People
  $E=\{(x,y) \mid x \text{ loves } y\}$

A Directed Graph



Types of Graphs: Summary

- Summary of the book's definitions.
- Keep in mind this terminology is not fully standardized across different authors...

Term	Edge type	Multiple edges ok?	Self-loops ok?
Simple graph	Undir.	No	No
Multigraph	Undir.	Yes	No
Pseudograph	Undir.	Yes	Yes
Directed graph	Directed	No	Yes
Directed multigraph	Directed	Yes	Yes

Adjacent Vertices (Neighbors)

Two vertices, *u* and *v* in an undirected graph *G* are called adjacent (or neighbors) in *G*, if {*u*, *v*} is an edge of *G*.

An edge *e* connecting *u* and *v* is called incident with vertices *u* and *v*, or is said to connect *u* and *v*.

The vertices *u* and *v* are called endpoints of edge {*u*, *v*}.

Directed Adjacency

- Let *G* be a directed (possibly multi-) graph
- Let *e* be an edge of *G* that is (or maps to) (*u*,*v*).

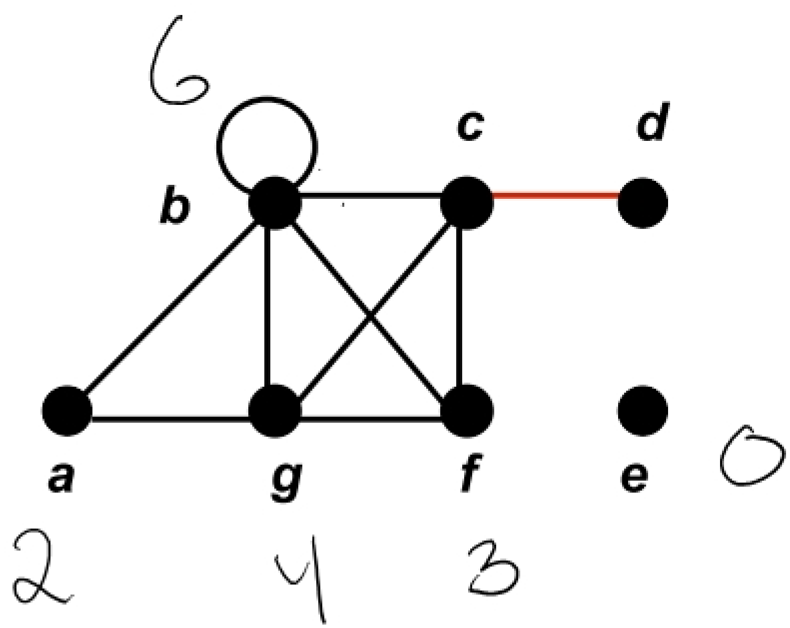
Then we say:

- *u* is *adjacent to v*, *v* is *adjacent from u*
- *e comes from* u, e *goes to* v.
- *e connects u to v*, *e goes from u to v*
- the *initial vertex* of *e* is *u*
- the *terminal vertex* of *e* is *v*

Degree of a vertex

The degree of a vertex in an undirected graph is

- the number of edges incident with it, except that
- a loop at a vertex contributes twice to the degree of that vertex



Directed Degree

- Let *G* be a directed graph, *v* a vertex of *G*.
  - The in-degree of v, $deg^{-}(v)$, is the number of edges going to v.
  - The out-degree of v, $deg^{+}(v)$, is the number of edges coming from v.
  - The degree of v, $deg(v) = deg^{-}(v)+deg^{+}(v)$, is the sum of v's in-degree and out-degree.

Handshaking Theorem

Let G = (V, E) be an undirected graph G with e edges. Then

$$\sum_{v \in V} \deg(v) = 2|E|$$

"The sum of the degrees over all the vertices equals

____"

NOTE: This applies even if multiple edges and loops are present.

Handshaking Theorem

- **Corollary:** Any undirected graph has an
even # of
vertices of odd degree.

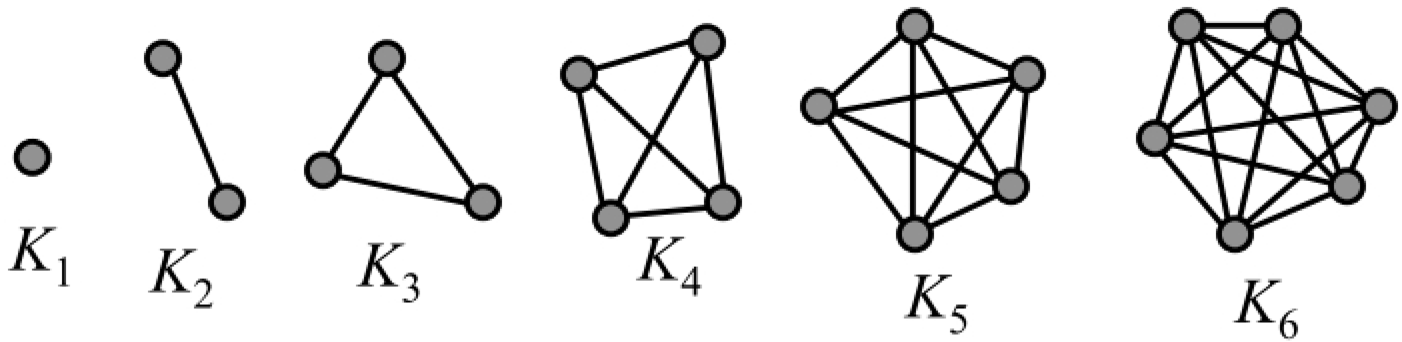
Special Graph Structures

Special cases of undirected graph structures:

- Complete graphs K_n
- Cycles C_n
- Wheels W_n
- *n*-Cubes Q_n
- Bipartite graphs
- Complete bipartite graphs $K_{m,n}$

Complete Graphs

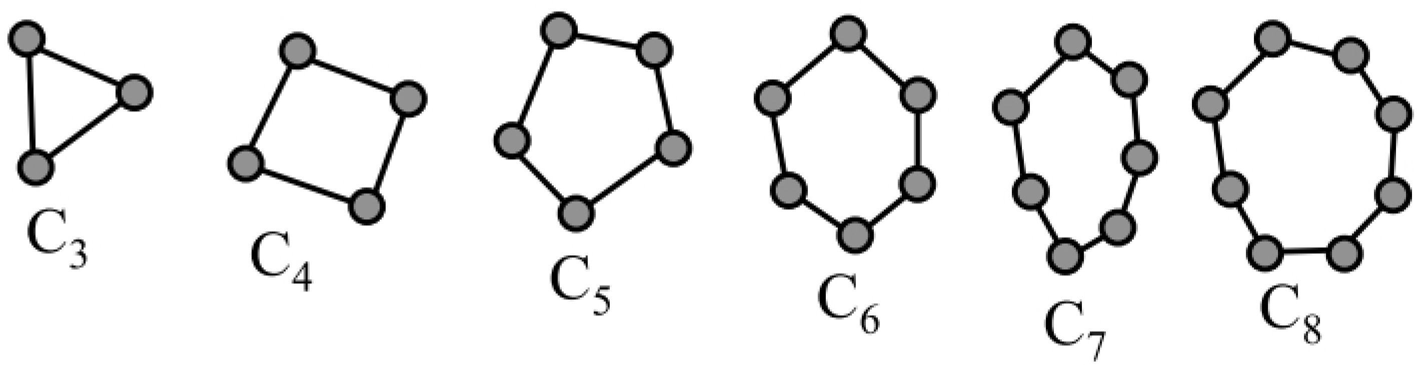
- For any $n \in \mathbf{N}$, a *complete graph* on *n* vertices, K_n, is a simple graph with *n* nodes in which every node is adjacent to every other node: $\forall u,v \in V: u \neq v \leftrightarrow \{u,v\} \in E$.



- How many edges does K_n have?

Cycles

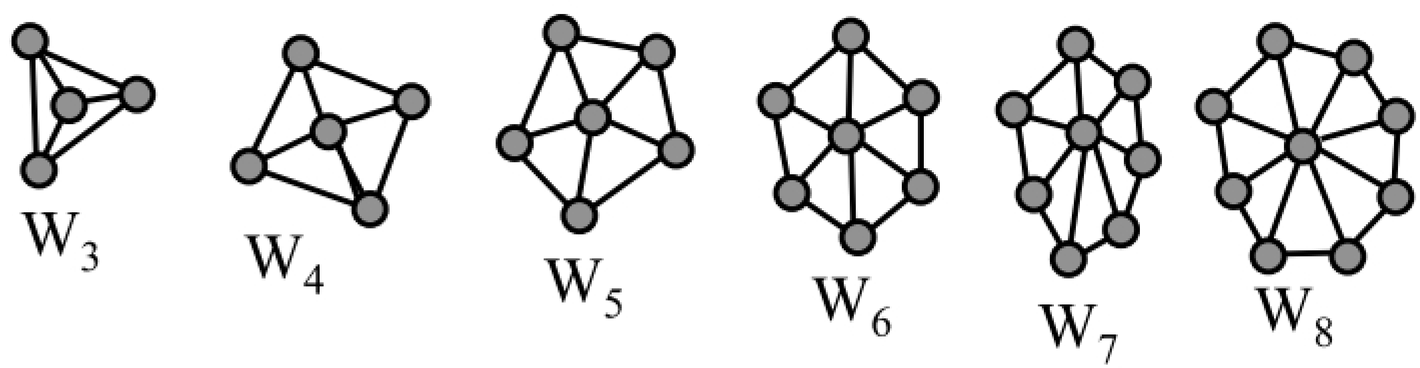
- For any $n \geq 3$, a *cycle* on *n* vertices, C_n, is a simple graph where $V=\{v_1, v_2, \ldots, v_n\}$ and $E=\{\{v_1,v_2\},\{v_2,v_3\},\ldots,\{v_{n-1},v_n\},\{v_n,v_1\}\}$.



How many edges are there in C_n?

Wheels

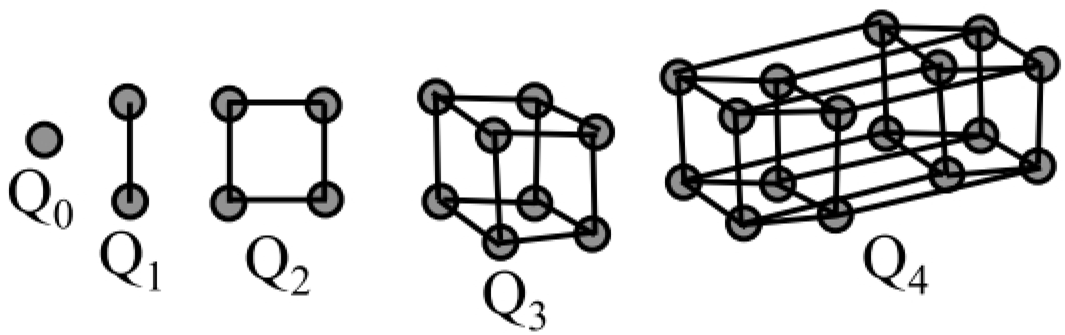
- For any $n \geq 3$, a *wheel* W_n, is a simple graph obtained by taking the cycle C_n and adding one extra vertex v_{hub} and n extra edges $\{\{v_{hub}, v_1\}, \{v_{hub}, v_2\}, \ldots, \{v_{hub}, v_n\}\}$.



How many edges are there in W_n?

n-cubes (hypercubes)

- For any $n \in \mathbf{N}$, the hypercube Q_n is a simple graph consisting of two copies of Q_{n-1} connected together at corresponding nodes. Q_0 has 1 node.



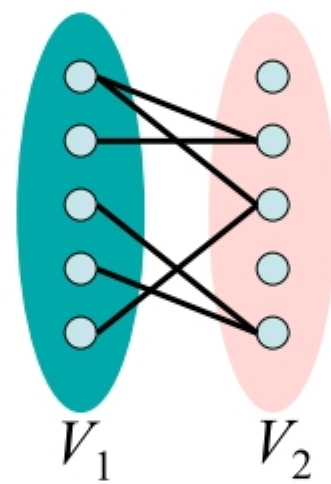
Number of vertices: 2^n. Number of edges:Exercise to try!

Bipartite Graphs

- **Def'n.:** A graph $G=(V,E)$ is *bipartite* (two-part) iff
  - $V = V_1 \cup V_2$
  - $V_1 \cap V_2 = \emptyset$

    $\forall e \in E: \exists v_1 \in V_1, v_2 \in V_2: e=\{v_1, v_2\}$.

- The graph canbe divided into two parts in such a way that all edges go between the two parts.



Complete Bipartite Graphs

- For $m,n \in \mathbf{N}$, the *complete bipartite graph* $K_{m,n}$ is a bipartite graph where

  $|V_1| = m$

  $|V_2| = n$

  and $E = \{\{v_1,v_2\} | v_1 \in V_1 \wedge v_2 \in V_2\}$.

$K_{m,n}$ has _____ nodes and _____ edges.