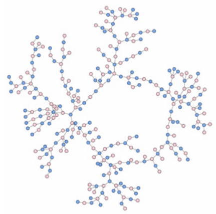

CS 173: Discrete Structures

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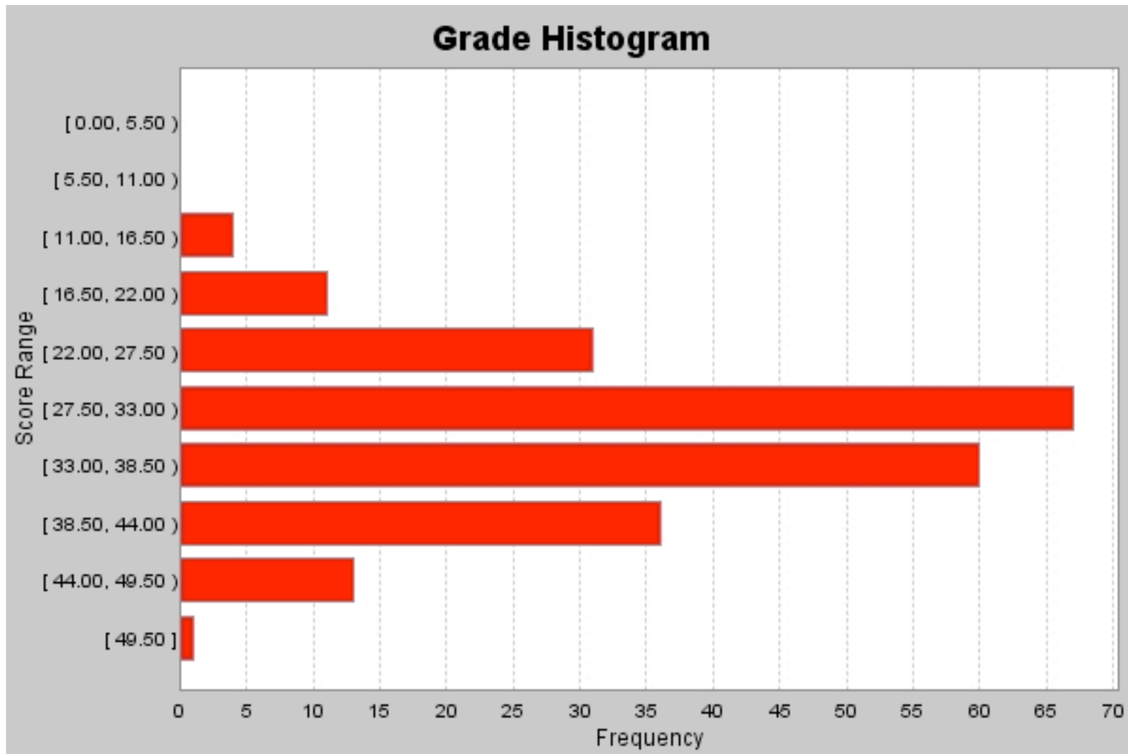
Announcements

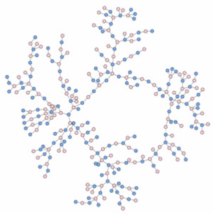
- Exam Results: 223 Exams
- Median Score: 32.5/50
- Do check your score and make sure we added correctly, etc.
- Approximate grades
  - >= 39 A
  - >= 33 B
  - >= 27.5 C
  - >= 21.5 D
  - >= 20 D-/F border
  - < 20 F
-



Announcements

- Exam Results:
  - Well…the exam was too long
  - You should look at this as an opportunity
    - You can figure what you didn't know, be prepared for final
  - Specifically:
    - Know how to solve a recurrence by unrolling
    - Be able to read pseudo-code
    - Be able to analyze the efficiency of an algorithm
    - Be able to state the Inductive Hypothesis of a given proof

- Do check your score and make sure we added correctly, etc.



Today…probability

- Sections 6.2 and 6.4 of the book
- What you should know

- We will focus solely on finite Sample Spaces

- Probability (single and combinations of events)
- Conditional Probability
- Are two events Independent?
- Probability Distributions
    - Uniform
    - Binomial
- Expectation
    - Linearity

Probability

Sample Space S: set of possible outcomes of an experiment

Event E: subset of the sample space

Assuming each outcome is equally likely Pr(E) =________

Note that Pr(E) + Pr($\bar{E}$) = _________

And: Pr(E1) ∪ Pr(E2) =______________________

Birthdays

How many people have to be in a room to assure that the probability that at least two of them have the same birthday is greater than 1/2?

a) 23
b) 183
c) 365
d) 730

Birthdays

How many people have to be in a room to assure that the probability that at least two of them have the same birthday is greater than 1/2?

Let p_n be the probability that no people share a birthday among n people in a room.

Then $1 - p_n$ is the probability that 2 or more share a birthday.

We want the smallest n so that $1 - p_n > 1/2$.

Birthdays

How many people have to be in a room to assure that the probability that at least two of them have the same birthday is greater than 1/2?

Assume 366 days in a year

Probability jth person's birthday does not match the j-1 prior birthdays is (366 - (j-1))/366= (367-j)/366

Probability of no matching birthdays:

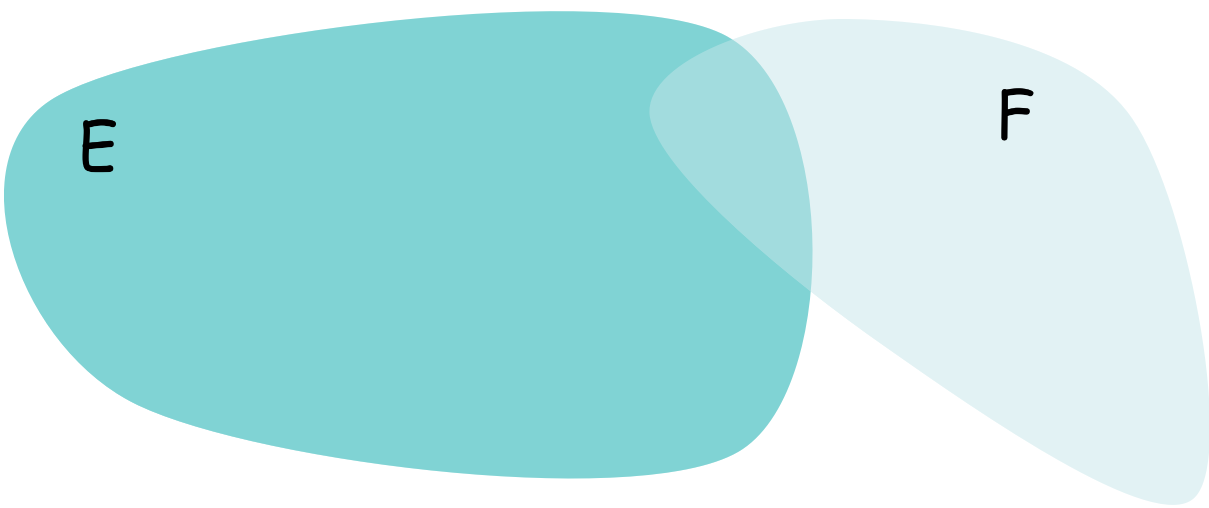
P_n= (365/366)(364/366)...((367-n)/366)

1 - P_n = 0.506 for n = 23

Conditional Probability

Let E and F be events with $\Pr(F) > 0$. The conditional probability of E given F, denoted by $\Pr(E|F)$ is defined to be:

$$\Pr(E|F) = \Pr(E \cap F)/\Pr(F).$$



Conditional Probability

$$Pr(E|F) = Pr(E \cap F)/Pr(F).$$

A bit string of length 4 is generated at random so that each of the 16 bit strings is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

$Pr(F) = 1/2$

$Pr(E \cap F)$? 0000 0001 0010 0011 0100

$Pr(E \cap F) = 5/16$ $Pr(E|F) = 5/8$

Conditional Probability

$$Pr(E|F) = Pr(E \cap F)/Pr(F).$$

What is the conditional probability that a family with 2 children has 2 boys, given that they have one boy

$Pr(F) =$

$Pr(E \cap F) =$

$Pr(E|F) =$

Independence

The events E and F are *independent* if and only if
$Pr(E \cap F) = Pr(E) \times Pr(F)$.

Let E be the event that a family of n children has children of both sexes.
Lef F be the event that a family of n children has at most one boy.

Are E and F independent if

$n = 2$?

Independence

The events E and F are *independent* if and only if
$Pr(E \cap F) = Pr(E) \times Pr(F)$.

Let E be the event that a family of n children has children of both sexes.

Lef F be the event that a family of n children has at most one boy.

Are E and F independent if

n = 3?

Independence

The events E and F are *independent* if and only if $Pr(E \cap F) = Pr(E) \times Pr(F)$.

Consider throwing 2 dice. Let E be getting a 4 on the first die and F getting 4 on the second die.

Bernoulli Trials

A "Bernoulli Trial" is an experiment with two outcomes: success (probability p) and failure (probability q) with p+q=1

Example: A coin is tossed 8 times. What is the probability of exactly 3 heads in the 8 tosses?

THHTTHTT is a tossing sequence…

How many ways of choosing 3 positions for the heads? $C(8,3)$

What is the probability of a particular sequence? $.5^8$

In general: The probability of exactly k successes in n independent Bernoulli trials with probability of success p, is $C(n,k)p^k(1-p)^{n-k}$

Bernoulli Trials

A game is played 5 times. Your chance of winning a single game is 0.70 . What is the probability that you win a majority of the 5 games?

What is the probability the the result is WWLLW?

$.7^3 .3^2$

Assumes independent trials

In general: The probability of exactly k successes in n independent Bernoulli trials with probability of success p, is $C(n,k)p^k(1-p)^{n-k}$

$$C(5,3)0.7^3 0.3^2 + C(5,4)0.7^4 0.3^1 + C(5,5)0.7^5 0.3^0$$

Binomial Distribution

In general: The probability of exactly k successes in n independent Bernoulli trials with probability of success p, is

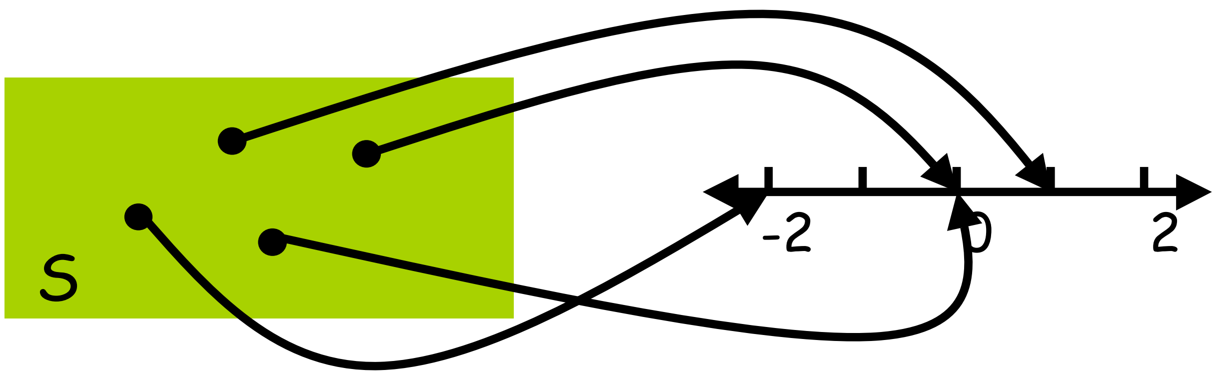
$$b(k;n,p) = C(n,k)p^{k}(1-p)^{n-k}$$

Considered as a function of k, this is called the

Binomial Distribution

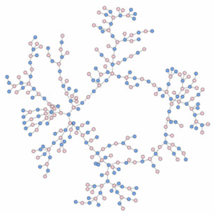
Random Variables

For a given sample space S, a *random variable* is any real valued function on S.



Suppose our experiment is a roll of 2 dice. S is set of pairs.

- X = sum of two dice. X((2,3)) = 5
- Y = difference between two dice. Y((2,3)) = 1
- Z = max of two dice. Z((2,3)) = 3



Random Variables

A probability distribution on a r.v. X is just an allocation of the total probability, 1, over the possible values of X.

How many movies have you watched in the last week?

a) 0
b) 1
c) 2
d) 3
e) 4

Picture gives a probability distribution!

The chart gives likelihood that a randomly selected student watched each of the particular numbers of movies.

Random Variables

Example: Do you ever play the game Racko?

Suppose we are playing a game with cards labeled 1 to 20, and we draw 3 cards. We bet that the maximum card has value 17 or greater. What's the probability we win the bet?

Let r.v. X denote the maximum card value. The possible values for X are 3, 4, 5, …, 20.

i	3	4	5	6	7	8	9	…	20
Pr(X = i)	?	?	?	?	?	?	?		?

Filling in this box would be a pain. We look for a general formula.

Random Variables

X is value of the highest card among the 3 selected. 20 cards are labeled 1 through 20.

We want Pr(X = i), i = 3,…20.

Denominator first: How many ways are there to select the 3 cards?

C(20,3)

How many choices are there that result in a max card whose value is i?

C(i-1,2)

Pr(X = i) = C(i-1, 2) / C(20,3)

These are the table values.

We win the bet if the max card, X is 17 or greater. What's the probability we win?

Pr(X = 17) + Pr(X = 18) + Pr(X = 19) + Pr(X = 20)

≈0.51

Expected Value

Let X be a discrete r.v. with set of possible values D. The *expected value* of X is:

$$E(X) = \sum_{x \in D} x \cdot \Pr(X = x)$$

Measure of central tendency.

Let X denote your score on the coming final. Suppose I assign scores according to the following distribution:

i	55	65	80	90
Pr(X=i)	0.1	0.3	0.4	0.2

Then E(X) = (55)(0.1) + 65(0.3) + (80)(0.4) + 90(0.2) = 75

Expected Value

Let X be a binomial r.v. with parameters n and p.

That is, X is the number of "successes" on n trials where each trial has probability of success p.

What is E(X)?

First we need $Pr(X = k) = C(n, k)\, p^k (1-p)^{n-k}$

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

Expected Value

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \frac{k \cdot n!}{(n-k)!k!} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!(k-1)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n} \frac{n-1!}{(n-k)!(k-1)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n} \binom{n-1}{k} p^{k-1} (1-p)^{n-k}$$

$$= np[p + (1-p)]^{n-1} \quad = np$$

Expected Value

Let X_i, i= 1,2,…,n, be a sequence of random variables, and suppose we are interested in their sum. The sum is a random variable itself with expectation given by:

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

The proof of this is inductive and algebraic. You can find it in your book on page 382.

Expected Value

Suppose you all (n) put your cell phones in a pile in the middle of the room, and I return them randomly. What is the expected number of students who receive their own phone back?

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

Define for i = 1, … n, a random variable:

$$X_i = \begin{cases} 1 & \text{if student } i \text{ gets the right phone,} \\ 0 & \text{otherwise.} \end{cases}$$

k	0	1
$Pr(X_i=k)$	1-(1/n)	1/n

$E[X_i]$ =

a) 1/n
b) 1/2
c) 1
d) No clue

$E[X_i] = Pr(X_i = 1)$

Expected Value

Suppose you all (n) put your cell phones in a pile in the middle of the room, and I return them randomly. What is the expected number of students who receive their own phone back?

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Define r.v. $X = X_1 + X_2 + \ldots + X_n$, and we want E[X].

$E[X] = E[X_1 + X_2 + \ldots + X_n]$

$= E[X_1] + E[X_2] + \ldots + E[X_n]$

$= 1/n + 1/n + \ldots + 1/n$

$= 1$

Expected Value

Suppose there are N couples at a party, and suppose m people get sleepy and leave. What is the expected number of couples left?

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

Define for i = 1, … N, a random variable:

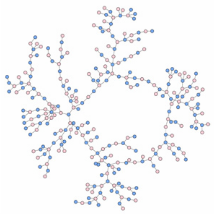
$$X_i = \begin{cases} 1 & \text{if couple i remains,} \\ 0 & \text{otherwise.} \end{cases}$$

Define r.v. $X = X_1 + X_2 + \ldots + X_n$, and we want E[X].

$E[X] = E[X_1 + X_2 + \ldots + X_n]$

$= E[X_1] + E[X_2] + \ldots + E[X_n]$

So what do we know about X_i?



Expected Value

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$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

Define for i = 1, … N, a random variable:

$$X_i = \begin{cases} 1 & \text{if couple i remains,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_i] = Pr(X_i = 1)$

(# of ways of choosing m from everyone else) /(# of ways of choosing m from all)

$$= \frac{\binom{2N-2}{m}}{\binom{2N}{m}}$$

$E[X_1] + E[X_2] + \ldots + E[X_n]$

$= n \times E[X_1] = (2N-m)(2N-m-1)/2(2N-1)$