CS 173: Discrete Structures

Eric Shaffer
Office Hour: Wed. 12-1, 2215 SC
shaffer1@illinois.edu

Combinations with repetition

$r = 10$

$n = 4$

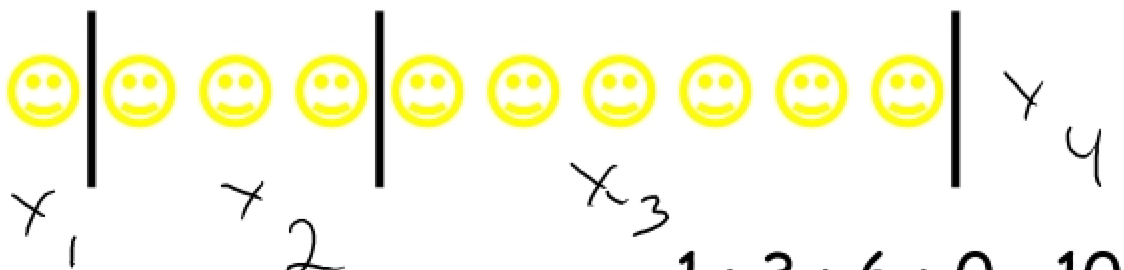
There are C(r+n-1,r) r-sized combinations from a set of n elements when repetition is allowed.

$$\binom{r+n-1}{r} = \binom{13}{10}$$

Example: How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 10$$

When the variables are nonnegative integers?



$$1 + 3 + 6 + 0 = 10$$

Permutations with indistinguishable objects

How many different strings can be made from the letters in the word rat? 6

How many different strings can be made from the letters in the word egg? 3

$3 \cdot 2 \cdot 1 = 3!$

egg
geg
gge

Permutations with indistinguishable objects

How many different strings can be made from the letters naannoon ?

Key thoughts: 8 positions, 3 kinds of letters to place.

In how many ways can we place the ns? C(8,4), now 4 spots are left

In how many ways can we place the as? C(4,2), now 2 spots are left

In how many ways can we place the os? C(2,2), now 0 spots are left

$$\binom{8}{4}\binom{4}{2}\binom{2}{2} = \frac{8!}{4!4!}\cdot\frac{4!}{2!2!}\cdot\frac{2!}{2!0!} = \frac{8!}{4!2!2!}$$

4 n's
2 a's
2 o's

_1 _2 _3 _4 _5 _6 _7 _8

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

n_i = # identical letters of type i

Permutations with indistinguishable objects

How many distinct permutations are there of the letters in the word APALACHICOLA?

$$\frac{12!}{4!2!2!}$$

How many if the two Ls must appear together?

$$\frac{11!}{4!2!}$$

How many if the first letter must be an A?

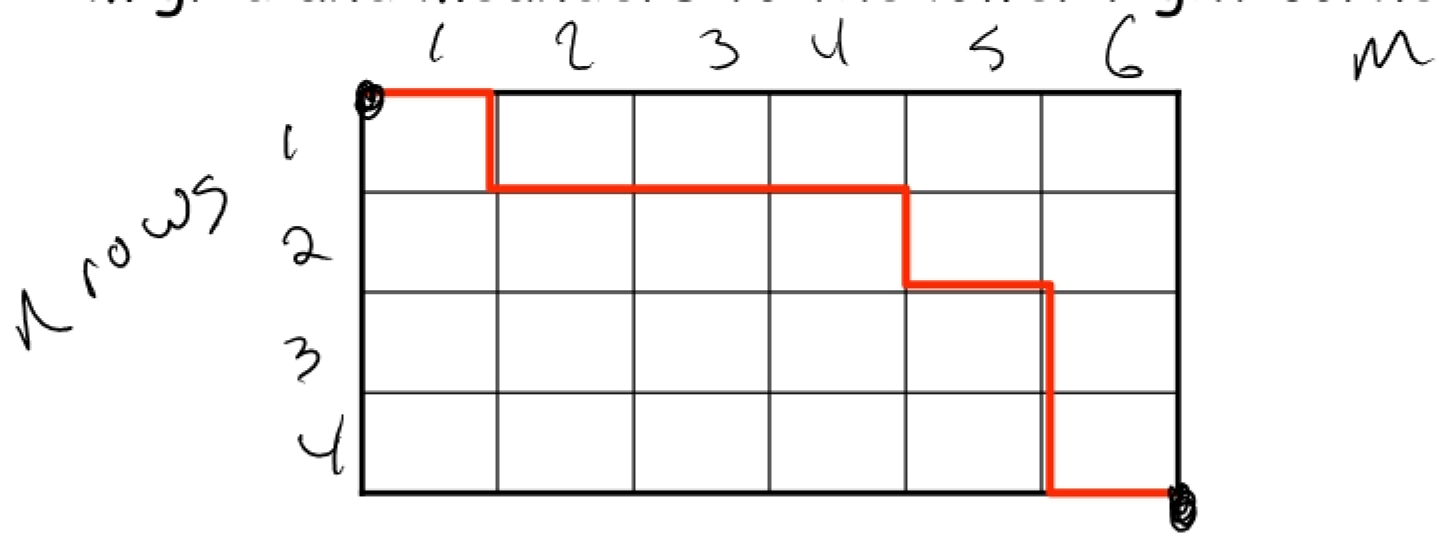
$$\frac{11!}{3!2!2!}$$

Walking the line

right move ≡ "R"
down ≡ "D"

A turtle begins at the upper left corner of an n x m grid and meanders to the lower right corner.

columns



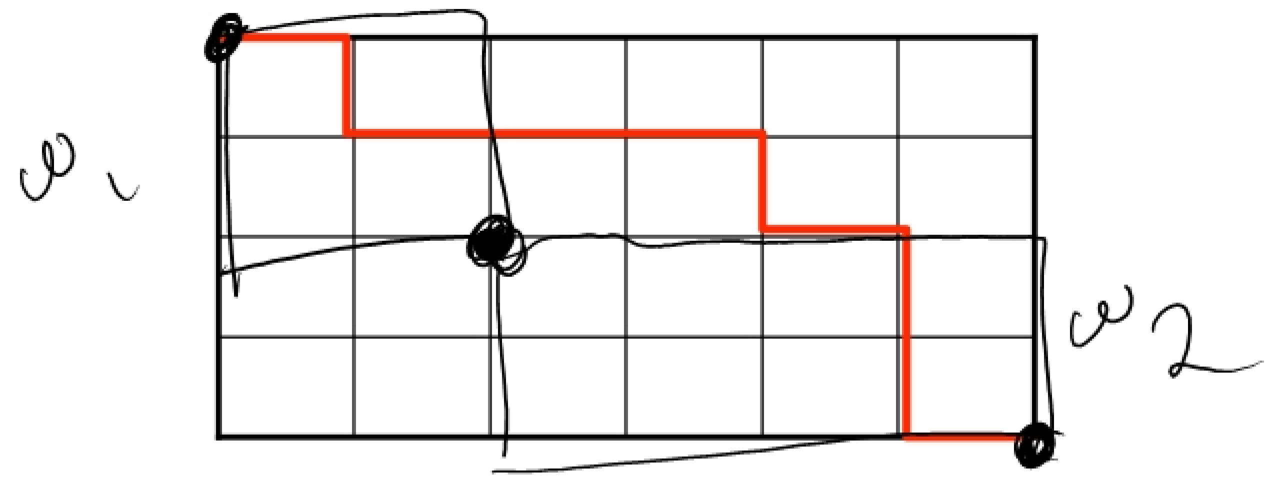
route is a string of m + n letters

How many routes could she take if she only moves right and down?

$$\binom{m+n}{m} = \binom{m+n}{n}$$

Walking the line

A turtle begins at the upper left corner of a m x n grid and meanders to the lower right corner.

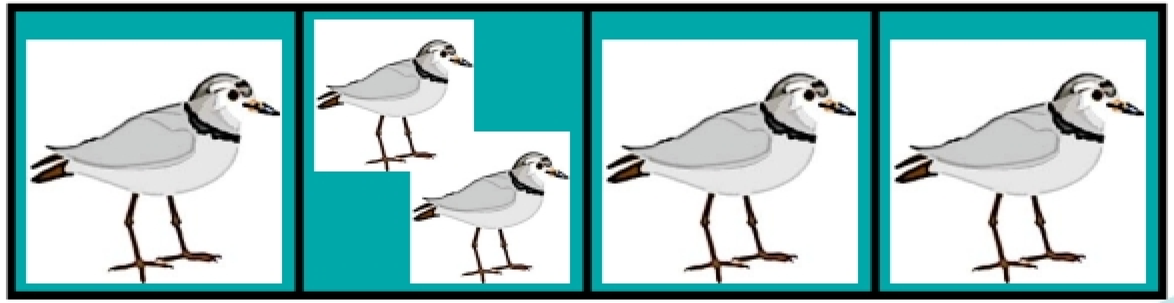


$\omega_1 \omega_2$

How many routes could she take if she only moves right and down, and if she **must** pass through the dot at point (a,b)?

Pigeonhole Principle

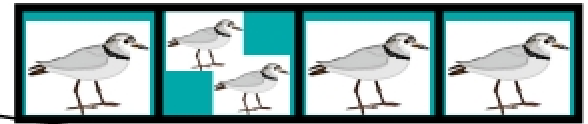
section 5.2



We can use this simple little fact to prove amazingly complex things.

If n pigeons fly into k pigeonholes and $k < n$, then some pigeonhole contains at least two pigeons.

Pigeonhole Principle



Let S contain any 6 positive integers. Then, there is a pair of numbers in S whose difference is divisible by 5.

Let $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Each of these has a remainder when divided by 5. What can these remainders be?

0, 1, 2, 3, or 4

6 numbers, 5 possible remainders…what do we know?

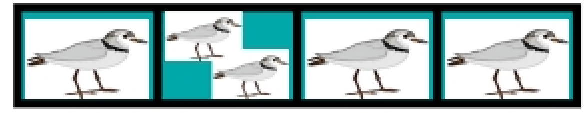
Some pair has the same remainder, by PHP.

Consider that pair, a_i and a_j, and their remainder r.

$a_i = 5m + r$, and $a_j = 5n + r$.

Their difference: $a_i - a_j = (5m + r) - (5n + r) = 5m - 5n = 5(m-n)$, which is divisible by 5.

Pigeonhole Principle

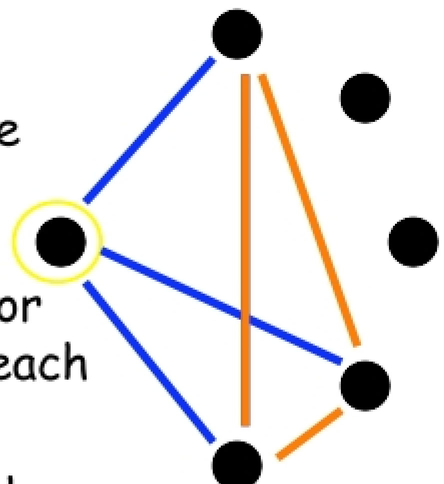


Six people go to a party. Either there is a group of 3 who all know each other, or there is a group of 3 who are all strangers.

Consider one person.

She either knows or doesn't know each other person.

But there are 5 other people! So, she knows, or doesn't know, at least 3 others.

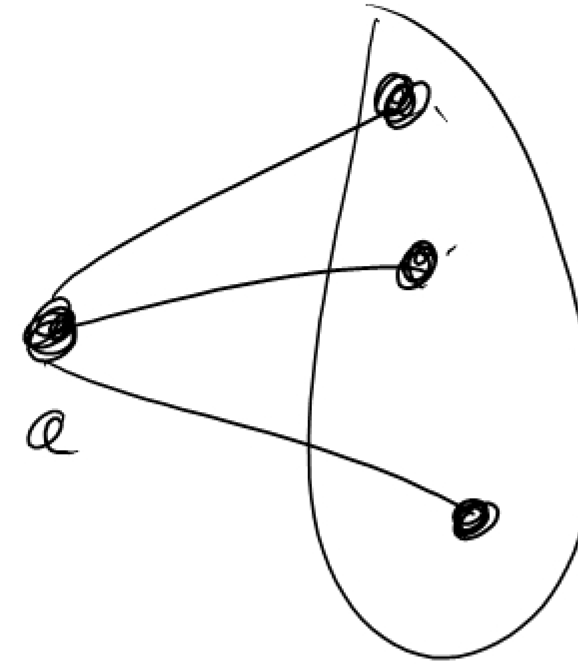


Let's say she knows 3 others.

If any of those 3 know each other, we have a blue △, which means 3 people know each other. So they all must be strangers.

But then we've proven our conjecture for this case.

The case where she *doesn't* know 3 others is similar.



Today…probability

- Sections 6.1 and 6.2 of the book
- Discrete probability

- Why should you study probability? Here's a few reasons:

    - Average case analysis of algorithms (CS 225)
    - Randomized algorithms and data structures (skip list)
    - Ability to excel in your fantasy baseball league

    - Network communication protocols
    - Spam filters
    - Computer graphics (monte carlo light transport)
    - Computational simulation in general…
    - Detecting plagiarism
    - Gambling (or "investing"), which is how it all started…

Great moments in Probability throughout history

1, 2, 4, 8

- 1654: French mathematicians Pascal and Fermat correspond to solve this problem: two players want to finish a dice game early and, given the current circumstances of the game, want to divide the stakes fairly based on the chance each has of winning the game from that point. Could you solve "The Problem of Points"?

- 1867: Dostoyevsky publishes one of his best and shortest novels, The Gambler, in order to pay off his own gambling debts. He played roulette using a martingale strategy (double your bet after every loss). Is this a good strategy?

Great moments in Probability throughout history

- 1962: Edward Thorp, American mathematician, publishes "Beat the Dealer", a book describing a winning Blackjack strategy. Thorp analyzed blackjack using Fortran programs he wrote on an IBM 704.
  - "Thorp became one of the very few applied mathematicians who risked physical harm in verifying a computer simulation." - Wikipedia
  - In 1974 founded Princeton/Newport Partners, the first(?) hedge fund, and proceeded to make a small fortune while helping establish the field of computational finance.

Probability

Which is more likely:

a) Rolling an 8 when 2 dice are rolled?
b) Rolling an 8 when 3 dice are rolled?
c) No clue.

Probability

Sample Space S: set of possible outcomes of an experiment

Event E: subset of the sample space

Assuming each outcome is equally likely Pr(E) = |E|/|S|

Note that $Pr(E) + Pr(\overline{E}) = 1$

And: $Pr(E1) \cup Pr(E2) = Pr(E1) + Pr(E2) - Pr(E1 \cap E2)$

"or"

1, 1 outcome
1, 2
⋮
6, 6

Probability

Let s be outcome in the Sample Space S

A function p(s) that assigns a probability to each $s \in S$ is called a Probability Distribution

Note: $0 \leq p(s) \leq 1$ and $\Sigma p(s_i) = 1$

We have been assuming that S is finite

It could also be a countably infinite set

It cannot have real-valued outcomes

(that requires continuous mathematics)

Probability

What is the probability of a total of 8 when 2 dice are rolled?

What is the size of the sample space? 36

How many rolls satisfy our condition of interest?

So the probability is $\frac{5}{36}$

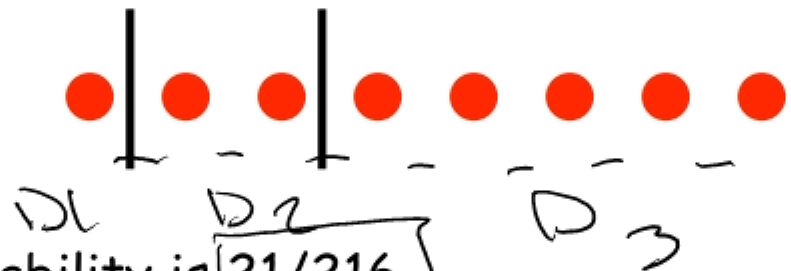
R	B
2	6
6	2
3	5
5	3
4	4

Probability

What is the probability of a total of 8 when 3 dice are rolled?

What is the size of the sample space? $6^3 = 216$

How many rolls satisfy our condition of interest? C(7,2)



So the probability is 21/216.

7 spots for 2 bars

$$\binom{7}{2} = \frac{7!}{5!\,2!} = 21$$

Lotteries

Pick 4: Win with the correct 4 digits in order

Pr(4of4)= $\frac{1}{10^4} = 0.0001$

Also a prize for 3 of 4 correct digits

Pr(3of4) = 9 possibilities for wrong digit · 4 spots $= \frac{36}{10^4} = 0.0036$

Sort of Powerball: Choose 6 numbers correctly from the first 40 integers

Pr(6) = $\frac{1}{\binom{40}{6}} = \frac{1}{3838380} \approx 0.00000026$

Poker

Deck has 52 cards, 4 “suits”, each suit has 13 “ranks”

A poker hand consists of 5 cards

“4 of a kind” means a hand with 4 cards of the same rank

Order is unimportant…..

Pr(4ofakind) =(# ways to choose rank x

ways to pick the 4 x

ways to pick the final card)/(total hands)

Pr(4ofakind) = C(13,1)C(4,4)C(48,1)/C(52,5)

=13(1)(48)/(2598960) = about 0.00024

Bitstrings

A string of 10 bits is randomly generated

What is the probability of it having at least one "0"?

One last example..

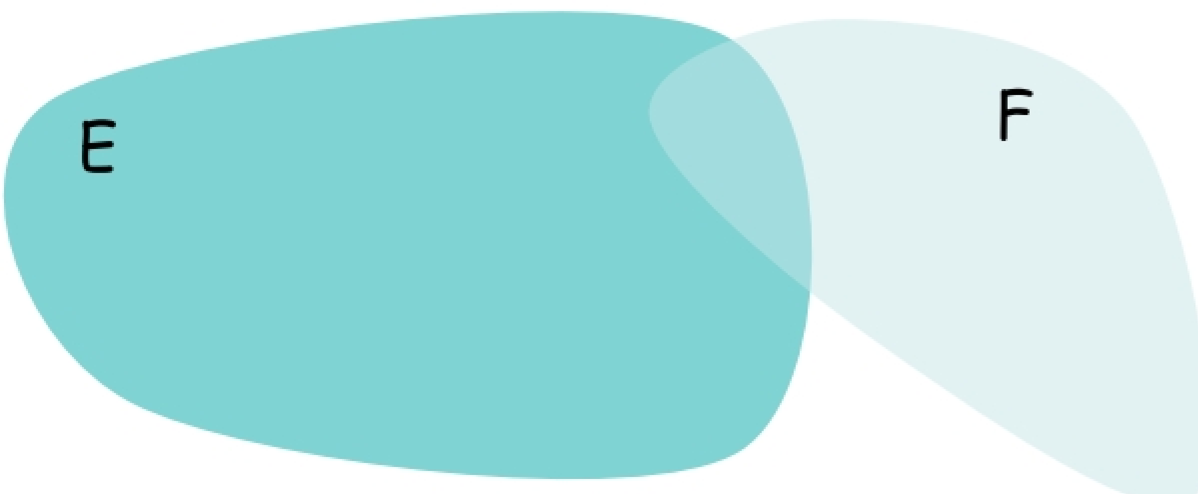
Pick an a positive integer not bigger than 100.

What is the probability it is divisible by 2 or 5?

Conditional Probability

Let E and F be events with $\Pr(F) > 0$. The conditional probability of E given F, denoted by $\Pr(E|F)$ is defined to be:

$$\Pr(E|F) = \Pr(E \cap F)/\Pr(F).$$



Conditional Probability

$$Pr(E|F) = Pr(E \cap F)/Pr(F).$$

A bit string of length 4 is generated at random so that each of the 16 bit strings is equally likely. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

$Pr(F) = 1/2$

$Pr(E \cap F)$? 0000 0001 0010 0011 0100

$Pr(E \cap F) = 5/16$ $Pr(E|F) = 5/8$

Independence

The events E and F are *independent* if and only if
$Pr(E \cap F) = Pr(E) \times Pr(F)$.

Let E be the event that a family of n children has children of both sexes.
Lef F be the event that a family of n children has at most one boy.

Are E and F independent if

$n = 2$? No

Independence

The events E and F are *independent* if and only if
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Let E be the event that a family of n children has children of both sexes.
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Are E and F independent if

$n = 3$? Yes

Independence

The events E and F are *independent* if and only if $Pr(E \cap F) = Pr(E) \times Pr(F)$.

Let E be the event that a family of n children has children of both sexes.

Lef F be the event that a family of n children has at most one boy.

Are E and F independent if

$n = 2$? No

$n = 3$? Yes

$n = 4$? No

$n = 5$? No

Bernoulli Trials

A coin is tossed 8 times. What is the probability of exactly 3 heads in the 8 tosses?

THHTTHTT is a tossing sequence…

How many ways of choosing 3 positions for the heads? $C(8,3)$

What is the probability of a particular sequence? $.5^8$

In general: The probability of exactly k successes in n independent Bernoulli trials with probability of success p, is $C(n,k)p^k(1-p)^{n-k}$

Bernoulli Trials

A game of Jewel Quest is played 5 times. You clear the board 70% of the time. What is the probability that you win a majority of the 5 games?

Sanity check: What is the probability the the result is WWLLW? $.7^3.3^2$ Assumes independent trials

In general: The probability of exactly k successes in n independent Bernoulli trials with probability of success p, is $C(n,k)p^k(1-p)^{n-k}$

$C(5,3)0.7^3 0.3^2 + C(5,4)0.7^4 0.3^1 + C(5,5)0.7^5 0.3^0$

Birthdays

How many people have to be in a room to assure that the probability that at least two of them have the same birthday is greater than 1/2?

a) 23
b) 183
c) 365
d) 730

Let p_n be the probability that no people share a birthday among n people in a room.

Then $1 - p_n$ is the probability that 2 or more share a birthday.

We want the smallest n so that $1 - p_n > 1/2$.

Birthdays

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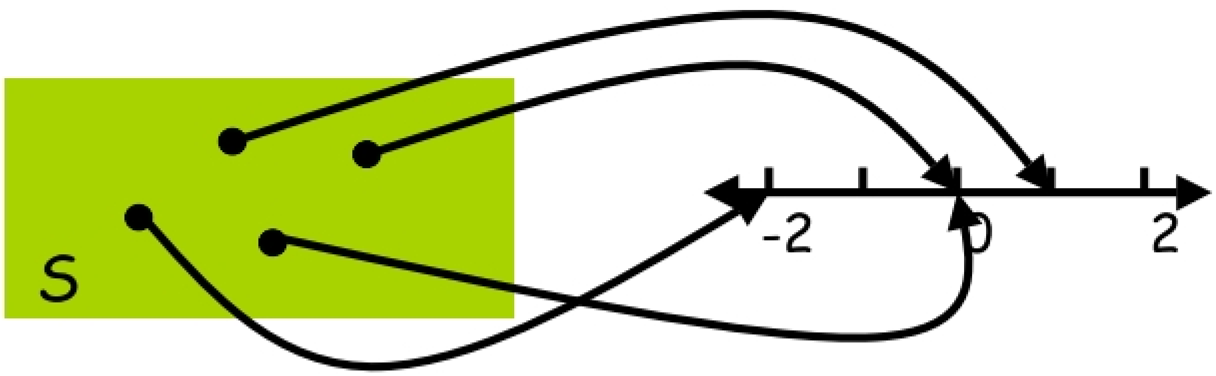
Random Variables

For a given sample space S, a *random variable* is any real valued function on S.



Suppose our experiment is a roll of 2 dice. S is set of pairs.

- X = sum of two dice. X((2,3)) = 5
- Y = difference between two dice. Y((2,3)) = 1
- Z = max of two dice. Z((2,3)) = 3

Random Variables

A probability distribution on a r.v. X is just an allocation of the total probability, 1, over the possible values of X.

How many movies have you watched in the last week?

a) 0
b) 1
c) 2
d) 3
e) 4

Picture gives a probability distribution!

The chart gives likelihood that a randomly selected student watched each of the particular numbers of movies.

Random Variables

Example: Do you ever play the game Racko?

Suppose we are playing a game with cards labeled 1 to 20, and we draw 3 cards. We bet that the maximum card has value 17 or greater. What's the probability we win the bet?

Let r.v. X denote the maximum card value. The possible values for X are 3, 4, 5, …, 20.

i	3	4	5	6	7	8	9	…	20
Pr(X = i)	?	?	?	?	?	?	?		?

Filling in this box would be a pain. We look for a general formula.

Random Variables

X is value of the highest card among the 3 selected. 20 cards are labeled 1 through 20.

We want Pr(X = i), i = 3,…20.

Denominator first: How many ways are there to select the 3 cards? C(20,3)

a) 20
b) 6840
c) 60
d) 1140
e) I'm not telling.

How many choices are there that result in a max card whose value is i? C(i-1,2)

Pr(X = i) = C(i-1, 2) / C(20,3) These are the table values.

We win the bet if the max card, X is 17 or greater. What's the probability we win?

Pr(X = 17) + Pr(X = 18) + Pr(X = 19) + Pr(X = 20)

≈0.51

Expected Value

Let X be a discrete r.v. with set of possible values D. The *expected value* of X is:

$$E(X) = \sum_{x \in D} x \cdot \Pr(X = x)$$

Measure of central tendency.

Let X denote your score on the coming midterm. Suppose I assign scores according to the following distribution:

i	55	65	80	90
Pr(X=i)	0.1	0.3	0.4	0.2

Then E(X) = (55)(0.1) + 65(0.3) + (80)(0.4) + 90(0.2) = 75

Expected Value

Let X be a binomial r.v. with parameters n and p.

That is, X is the number of "successes" on n trials where each trial has probability of success p.

Defn of Binomial Distribution.

What is E(X)?

First we need $\Pr(X = k) = C(n, k)\, p^k (1-p)^{n-k}$

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

Expected Value

$$E(X) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \frac{k \cdot n!}{(n-k)!k!} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!(k-1)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n} \frac{n-1!}{(n-k)!(k-1)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n} \binom{n-1}{k} p^{k-1} (1-p)^{n-k}$$

$$= np[p + (1-p)]^{n-1} \quad = np$$

Expected Value

Let X_i, i= 1,2,…,n, be a sequence of random variables, and suppose we are interested in their sum. The sum is a random variable itself with expectation given by:

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

The proof of this is inductive and algebraic. You can find it in your book on page 382.

Expected Value

Suppose you all (n) put your cell phones in a pile in the middle of the room, and I return them randomly. What is the expected number of students who receive their own phone back?

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

Define for i = 1, … n, a random variable:

$$X_i = \begin{cases} 1 & \text{if student } i \text{ gets the right phone,} \\ 0 & \text{otherwise.} \end{cases}$$

k	0	1
$Pr(X_i=k)$	1-(1/n)	1/n

$E[X_i] =$

a) 1/n
b) 1/2
c) 1
d) No clue

$E[X_i] = Pr(X_i = 1)$

Expected Value

Suppose you all (n) put your cell phones in a pile in the middle of the room, and I return them randomly. What is the expected number of students who receive their own phone back?

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Define r.v. $X = X_1 + X_2 + \ldots + X_n$, and we want $E[X]$.

$E[X] = E[X_1 + X_2 + \ldots + X_n]$

$= E[X_1] + E[X_2] + \ldots + E[X_n]$

$= 1/n + 1/n + \ldots + 1/n$

$= 1$

Expected Value

Suppose there are N couples at a party, and suppose m people get sleepy and leave. What is the expected number of couples left?

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

Define for i = 1, … N, a random variable:

$$X_i = \begin{cases} 1 & \text{if couple i remains,} \\ 0 & \text{otherwise.} \end{cases}$$

Define r.v. $X = X_1 + X_2 + \ldots + X_n$, and we want $E[X]$.

$E[X] = E[X_1 + X_2 + \ldots + X_n]$

$= E[X_1] + E[X_2] + \ldots + E[X_n]$

So what do we know about X_i?

Expected Value

Suppose there are N couples at a party, and suppose m people get sleepy and leave. What is the expected number of couples left?

$$E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i]$$

Define for i = 1, … N, a random variable:

$$X_i = \begin{cases} 1 & \text{if couple i remains,} \\ 0 & \text{otherwise.} \end{cases}$$

$E[X_i] = Pr(X_i = 1)$ (# of ways of choosing m from everyone else) /(# of ways of choosing m from all)

$$= \frac{\binom{2N-2}{m}}{\binom{2N}{m}}$$

$E[X_1] + E[X_2] + … + E[X_n]$

$= n \times E[X_1] = (2N-m)(2N-m-1)/2(2N-1)$