CS 173: Discrete Structures

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Review

- For those who like to read ahead:
  - Lecture schedule on the web includes all future readings

- What you should know:
  - Section 5.1
    - Rule of Sum
    - Rule of Product
    - Tree diagrams
  - Section 5.3
    - Permutations
    - Combinations
- Today we'll cover sections 5.4 and 5.5

Permutations

In how many ways can 5 distinct labradoodles and 3 distinct rottweilers stand in line, if no two rottweilers stand together?

$5! = \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

rots $= P(6,3) = \frac{6}{r_1}\ \frac{5}{r_2}\ \frac{4}{r_3}$

___ L1 ___ L2 ___ L3 ___ L4 ___ L5 ___

s1 s2 s3 s4 s5 s6

$$\boxed{\underbrace{5!}_{\text{labs}} \cdot \frac{6!}{3!}}$$

5! X P(6,3)

Combinations

A combination is an unordered selection of elements from some set.

The number of combinations of r distinct objects chosen from n distinct objects is denoted by $C(n,r)$ or $\binom{n}{r}$, and is read "n choose r."

$$C(n,r) = P(n,r)/r! = \boxed{n!/((n-r)!r!)}$$

Combinations

unordered

$$\binom{12}{5} = \frac{12!}{7!\,5!}$$

A basketball team consists of 12 players, 5 of which make up the starters. How many different sets of starters can you make from the 12?

What's the difference ?

order is important

$$P(12,5) = \frac{12}{1}\cdot\frac{11}{2}\cdot\frac{10}{3}\cdot\frac{9}{4}\cdot\frac{8}{5}$$

In a running race of 12 sprinters, each of the top 5 finishers receives a different medal. How many ways are there to award the 5 medals?

$\boxed{P(12,5)}$ $\boxed{= C(12,5) \times 5!}$

$$= \frac{12!}{7!}$$

Combinations

A committee of 4 students is to be selected from a class consisting of 19 ece majors, and 34 cs majors

order **not** important

In how many ways can a committee with exactly 1 ece major be selected?

$$\binom{19}{1}\binom{34}{3} = 19\left(\frac{34!}{31!\,3!}\right)$$

In how many ways can a committee with at most 1 ece major be selected?

$$\binom{34}{4} + \boxed{}$$

c's w/ 1 ece

In how many ways can a committee with at least 1 ece major be selected?

$$\binom{19}{1}\binom{34}{3} + \binom{19}{2}\binom{34}{2} + \binom{19}{3}\binom{34}{1} + \binom{19}{4}$$

or $\boxed{\binom{53}{4} - \binom{34}{4}}$

Binomial Coefficients

A monomial is a product of powers of variables.

Like $\boxed{a^2b^3cd^5}$

A binomial is a sum of two monomials

Like $\boxed{a+b}$

A binomial raised to a power generates a polynomial

$(a + b)^2 = a^2 + 2ab + b^2$ ✓
$(a + b)^3 = a^3 + 3ab^2 + 3a^2b + b^3$
$(a + b)^4 = a^4 + 4ab^3 + 6a^2b^2 + 4a^3b + b^4$ ✓

Coefficients are the constants in front of the monomials

Like $\underbrace{1,4,6,4,1}$

Binomial Coefficients

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$\underbrace{(a + b)^4} = a^4 + \underbrace{4a^3b} + 6a^2b^2 + 4ab^3 + b^4$

Look at the sum of powers in each monomial.

Do you see a pattern?

a a a a

a a a b

a a b b

a b b b

b b b b

Binomial Coefficients

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Look at the sum of powers in each monomial.
For each term we get to choose 4 variables
Each possible combination is represented in the expansion $1 = \binom{4}{0}$ aaaa

We have 4 binomials we pick variables from: $4 = \binom{4}{1}$ aaab

$\underbrace{(a+b)(a+b)(a+b)(a+b)}$

There is only 1 way to get a term with a^4
There are 4 ways to get a term with a^3 $6 = \binom{4}{2}$ aabb
There are 6 ways to get a term with a^2

Binomial Coefficients

$(a + b)^4 = (a + b)(a + b)(a + b)(a + b)$

$$= \binom{4}{0} a^4 + \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 + \binom{4}{3} a b^3 + \binom{4}{4} b^4$$

Binomial Theorem: Let x and y be variables, and let n be any nonnegative integer. Then

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Binomial Coefficients

Binomial Theorem: Let x and y be variables, and let n be any nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

What is the coefficient of a^8b^9 in the expansion of $(3a+2b)^{17}$?

What is n? 17

What is j? 9

What is x? 3a

What is y? 2b

$$\binom{17}{9}(3a)^8(2b)^9 = \binom{17}{9}3^8 2^9 a^8 b^9$$

$$\boxed{\binom{17}{9}3^8 2^9}$$

Combinatorial proofs

$x_1 \quad x_2 \quad \cdots \quad x_n$

$0 \quad 1 \quad \quad 1$

$\underline{2} \cdot \underline{2} \cdots \underline{2}$

$= 2^n$

A "combinatorial proof" can be used to prove identities

- Done by counting the same thing in 2 different ways

As an example: $\boxed{\sum_{j=0}^{n} \binom{n}{j} = \underline{2^n}}$

How many subsets of are there of an n element set?

How many subsets of size 0 does it have? $\binom{n}{0} = 1$

How many subsets of size 1 does it have? $\binom{n}{1}$

How many subsets of size 2 does it have? $\binom{n}{2}$

$$\boxed{\sum_{j=0}^{n} \binom{n}{j} = 2^n}$$

Proofs using the Binomial Theorem

We can also use the binomial theorem to prove identities

$$2^n = (1+1)^n = \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} 1^j = \sum_{j=0}^{n} \binom{n}{j}$$

Let x=1 and y=1 in Binomial Theorem.

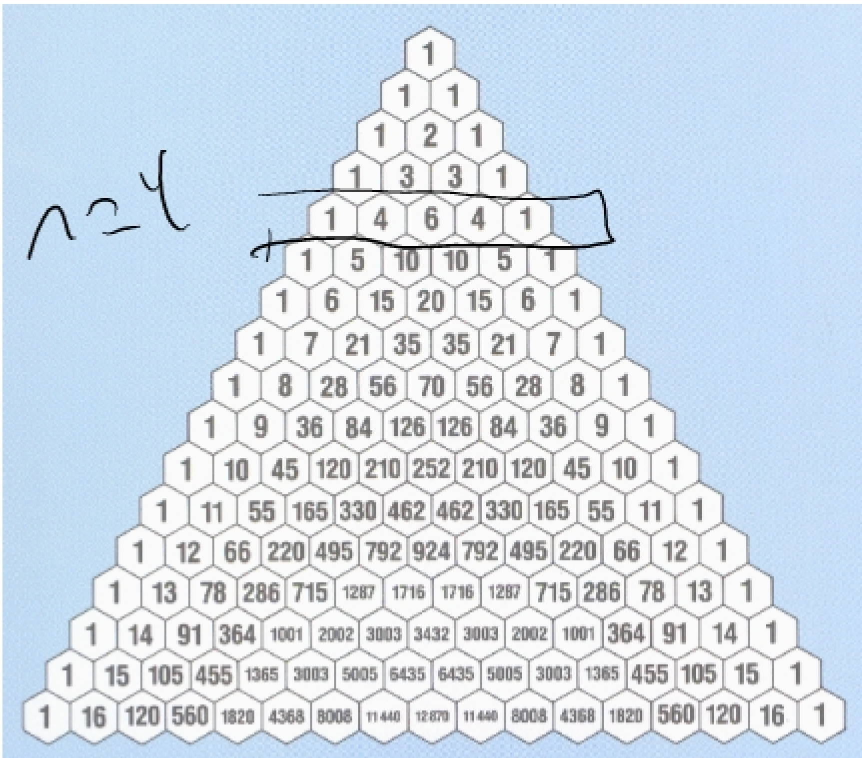
$$\sum_{j=0}^{n} \binom{n}{j} 1^{n-j} 1^j = (1+1)^n$$

$$\sum_{j=0}^{n} \binom{n}{j} = 2^n$$

$(a+b)^2$

Pascal's Triangle

Pascal's Triangle illustrates the pattern of binomial coefficients



$n=4$

$$\boxed{\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j}}$$

Combinatorial proof:

Consider a set S with n elements

Count # subsets with j elements $= \binom{n}{j} =$

Consider specific element "a" in S

How many |j| susbsets contain element "a" $\binom{n-1}{j-1}$ subsets size j with "a"

$+$

How many |j| subsets don't contain "a" $\binom{n-1}{j}$ subsets size j without "a"

What is the sum of each row of Pascal's Triangle?

For row n

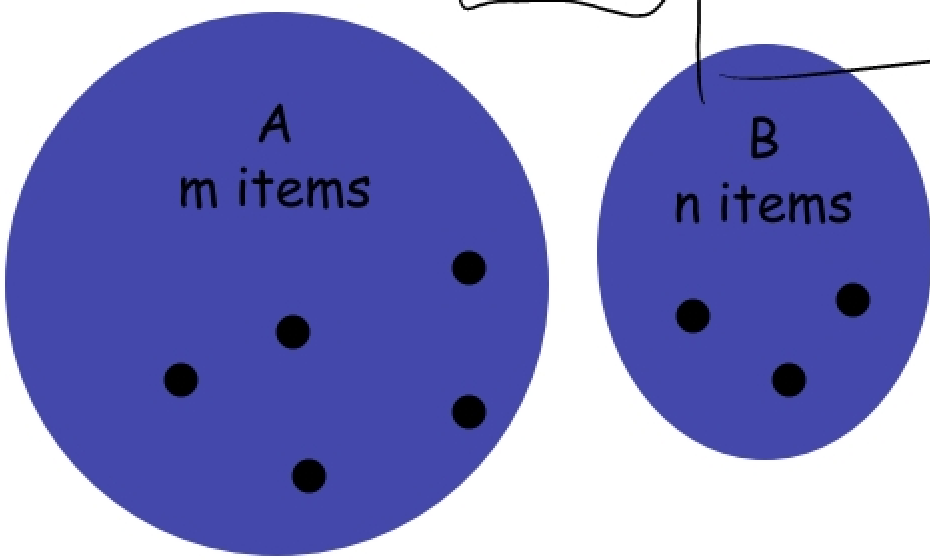
$$\boxed{2^n}$$

$n=0$ 1
$n=1$ 1 1
$n=2$ 1 2 1
$n=3$ 1 3 3 1
$n=4$ 1 4 6 4 1

Vandermonde's Identity

Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{j=0}^{r} \binom{m}{r-j}\binom{n}{j}$$



To choose r items, take some from A and some from B. All possible ways of doing this gives the result.

cases: choose 0 from B

$\binom{m}{r}\binom{n}{0}$ +

1 from B $\binom{m}{r-1}\binom{n}{1}$ +

$\vdots$

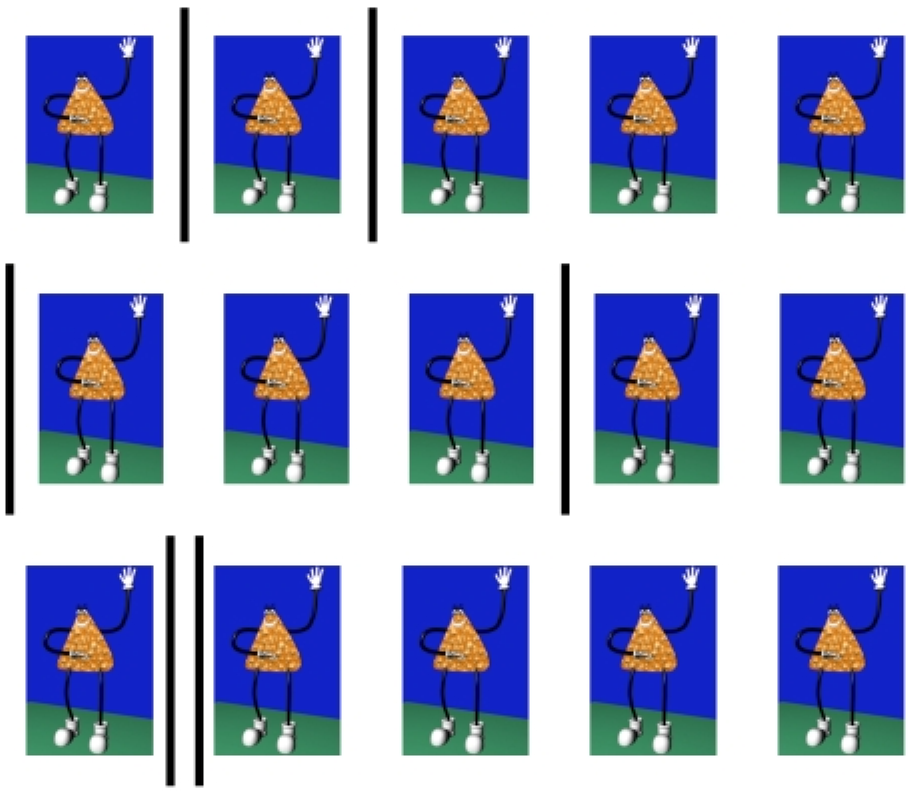
all from B $\binom{m}{0}\binom{n}{n}$ +

Combinations with repetition

Suppose you want to buy 5 bags of chips from the 3 kinds you like at Meijer. In how many different ways can you stock up?

$* \mid * \; * \mid * \; *$

bin 1 | bin 2 | bin 3



Out of 7 items, we are choosing 2 to be bars.

From that, and our understanding of the model, we can report the answer.

$$\binom{7}{2} = \boxed{\binom{7}{5}}$$

7 symbols

$\underset{s_1}{*} \quad \underset{s_2}{*} \quad \underset{s_3}{|} \quad \underset{s_4}{*} \quad \underset{s_5}{*} \quad \underset{s_6}{|} \quad \underset{s_7}{*}$

Combinations with repetition

If we pick r things (with repetition) from n bins

- The n bins have n-1 bars separating them
- The r things we'll draw as stars

A drawing of a specific selection will have n-1 bars and r stars

$\boxed{\text{n+r-1 symbols}}$

~~we choose~~ r symbols to stars

$\binom{n+r-1}{r}$ place the stars

There are C(n+r-1,r) ways to do this.

Combinations with repetition

$x_1 \mid x_2 \mid x_3 \mid x_4$

There are C(r+n-1,r) r-sized combinations from a set of n elements when repetition is allowed.

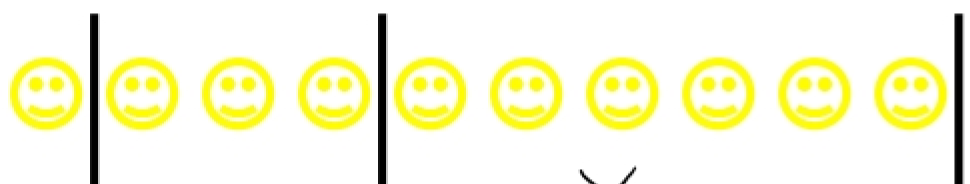
$$\binom{10+3}{10} = \binom{13}{10}$$

$r=10 \quad n=4$

Example: How many solutions are there to the equation

$$\underbrace{x_1 + x_2 + x_3 + x_4 = 10}$$

When the variables are nonnegative integers?



$x_1 \quad x_2 \quad x_3 \quad x_4$

$$\underbrace{1 + 3 + 6 + 0 = 10}$$

Permutations with indistinguishable objects

How many different strings can be made from the letters in the word rat? 6

How many different strings can be made from the letters in the word egg? 3

Permutations with indistinguishable objects

How many different strings can be made from the letters naannoon ?

Key thoughts: 8 positions, 3 kinds of letters to place.

In how many ways can we place the ns? $C(8,4)$, now 4 spots are left

In how many ways can we place the as? $C(4,2)$, now 2 spots are left

In how many ways can we place the os? $C(2,2)$, now 0 spots are left

$$\binom{8}{4}\binom{4}{2}\binom{2}{2} = \frac{8!}{4!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{2!0!} = \frac{8!}{4!2!2!}$$

Permutations with indistinguishable objects

How many distinct permutations are there of the letters in the word APALACHICOLA?

$$\frac{12!}{4!2!2!}$$

How many if the two Ls must appear together?

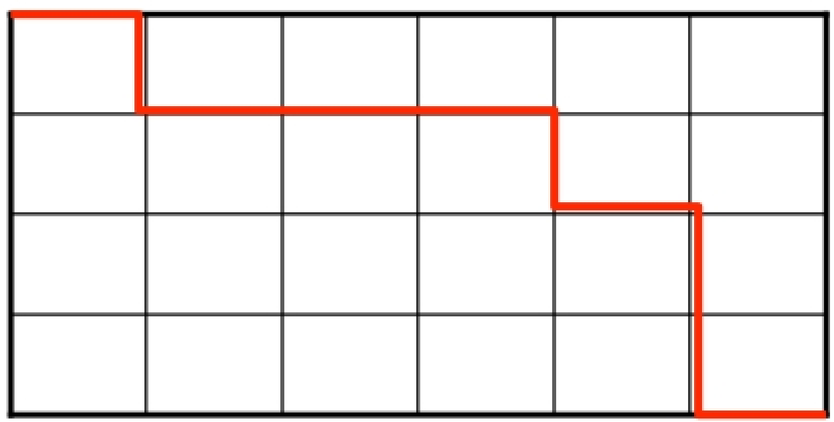
$$\frac{11!}{4!2!}$$

How many if the first letter must be an A?

$$\frac{11!}{3!2!2!}$$

Walking the line

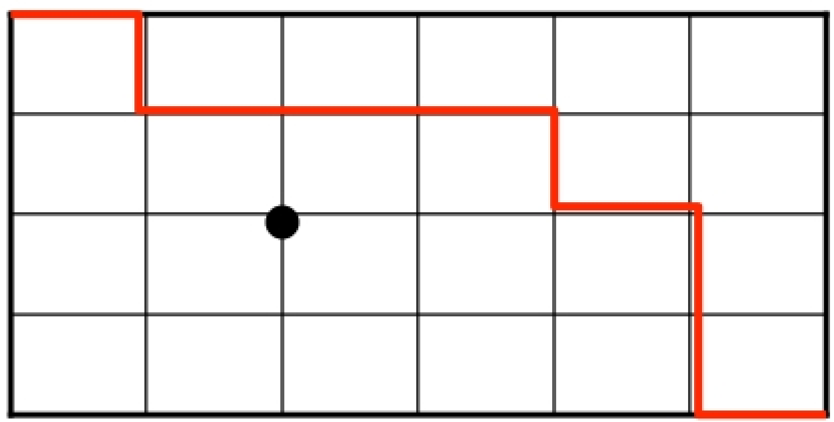
A turtle begins at the upper left corner of an n x m grid and meanders to the lower right corner.



How many routes could she take if she only moves right and down?

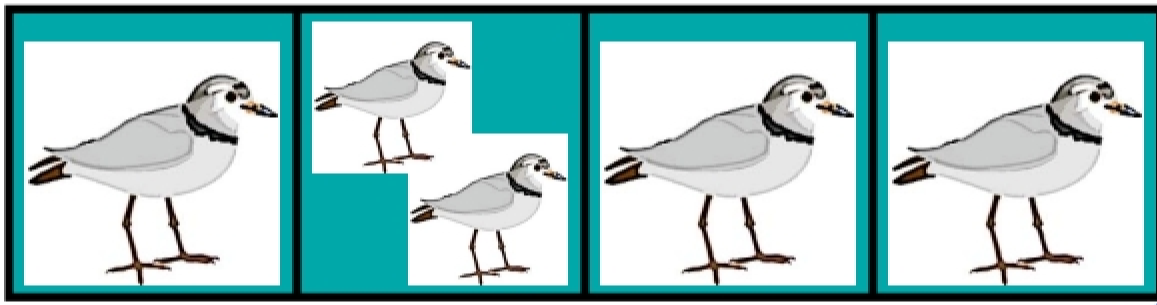
Walking the line

A turtle begins at the upper left corner of a m x n grid and meanders to the lower right corner.



How many routes could she take if she only moves right and down, and if she **must** pass through the dot at point (a,b)?

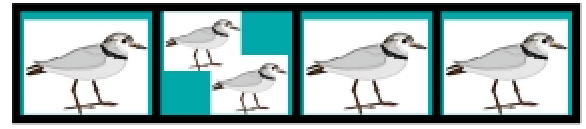
Pigeonhole Principle



We can use this simple little fact to prove amazingly complex things.

If n pigeons fly into k pigeonholes and k < n, then some pigeonhole contains at least two pigeons.

Pigeonhole Principle



Let S contain any 6 positive integers. Then, there is a pair of numbers in S whose difference is divisible by 5.

Let $S = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Each of these has a remainder when divided by 5. What can these remainders be?

0, 1, 2, 3, or 4

6 numbers, 5 possible remainders…what do we know?

Some pair has the same remainder, by PHP.

Consider that pair, a_i and a_j, and their remainder r.

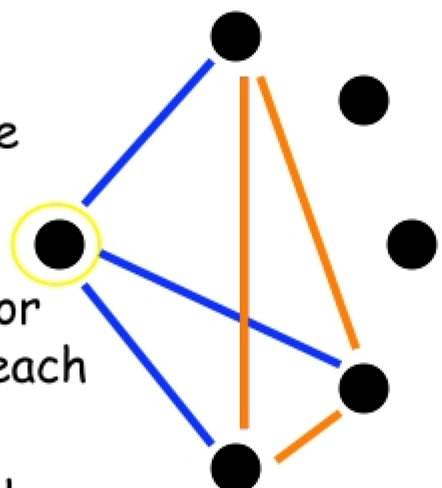
$a_i = 5m + r$, and $a_j = 5n + r$.

Their difference: $a_i - a_j = (5m + r) - (5n + r) = 5m - 5n = 5(m-n)$, which is divisible by 5.

Pigeonhole Principle

Six people go to a party. Either there is a group of 3 who all know each other, or there is a group of 3 who are all strangers.

Consider one person.



She either knows or doesn't know each other person.

But there are 5 other people! So, she knows, or doesn't know, at least 3 others.

Let's say she knows 3 others.

If any of those 3 know each other, we have a blue △, which means 3 people know each other. So they all must be strangers.

But then we've proven our conjecture for this case.

The case where she *doesn't* know 3 others is similar.