CS 173: Discrete Structures

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Announcements

- Friday 3/13 class cancelled (due to EOH)
- HW still due 3/13
  - Hand-in at 3229 Siebel at ~~before~~ noon at the latest

- Quiz **Next** Wed. 3/18 Covers through today's lecture
  - Further updates as we get closer to that date..

- Today:
  - Solving recurrence relations (Section 7.1)

Big-O review

- Which of these functions is $O(x^2)$?

1. $F(x) = 10$ yes
2. $F(x) = x \log x$ yes
3. $F(x) = 2^x$ no
4. $F(x) = x^4/2$ no
5. $F(x) = \lceil x \rceil \lfloor x \rfloor$ yes

$\lfloor x \rfloor \le \lceil x \rceil \le x+1$ so $\lceil x \rceil \lfloor x \rfloor \le (x+1)^2$

$= x^2 + 2x + 1 =$

$O(x^2)$

Fill-in-the-blank

If $\exists$ c,k so that $\forall$ x $\geq$ k, $f(x) \geq c\, g(x)$
then $f(x) = \Omega(g(x))$.

If $f(x) = O(g(x))$, and $f(x) = \Omega(g(x))$
then $f(x) = \Theta(g(x))$

Growth of functions - other estimates

Show that $3x^2+8x \log x$ is $\Theta(x^2)$

Proof:

1) $3x^2+8x \log x$ is $O(x^2)$ since $8x \log x \leq 8x^2$

   so $3x^2+8x \log x \leq 11x^2$ for $x>1$

2) $3x^2+8x \log x$ is $\underline{\Omega(x^2)}$

$$3x^2 + 8x \log x \geq x^2 \quad \text{for } x > 1$$

$\boxed{\text{i.e. } k=1,\ c=1}$

so $3x^2+8x\log x = \Omega(x^2)$

Growth of functions - other estimates

$\lg = \log_2$

Show that $2n \lg 2n$ is $\Theta(n \lg n^2)$ $\stackrel{?}{=}$ $2n(\lg 2 + \lg n) = \Theta(n 2 \lg n)$

Proof:

Show $2n\lg 2n = O(n\lg n^2)$ show $2n + 2n\lg n = \Theta(2n\lg n)$

Let $c = 2$ and $k = 1$

$2n + 2n\lg n \leq 2n\lg n + 2n\lg n = 2\cdot(2n\lg n)$

for $n > 1$ so $2n\lg 2n = O(n\lg n^2)$

Show $2n\lg 2n = \Omega\, n\lg n^2$

$2n + 2n\lg n \geq 2n\lg n$ for all $n > 1$

so is true.

Growth of functions - other estimates

For functions f and g, $f = o(g)$ if

$$\forall c>0\ \exists k \text{ so that } \forall\ n>k,\ f(n) \le c\cdot g(n),$$

"f is little-o of g"

Example: Show that $n^2 = o(n^2 \log n)$

Proof: find a k (possibly in terms of c) that makes the inequality hold.

Choose c arbitrarily. How large does n have to be so that $n^2 \le c\, n^2 \log n$?

$$1 \le c \log n$$

$$1/c \le \log n$$

$$2^{1/c} \le n$$

This inequality holds when $n > 2^{1/c}$.

So, $k = 2^{1/c}$.

Growth of functions - other estimates

For functions f and g, $f = o(g)$ if

$$\forall c>0\ \exists k \text{ so that } \forall\ n>k,\ f(n) \le c\cdot g(n),$$

"f is little-o of g"

Example: Show that $10n^2 = o(n^3)$

Proof foreshadowing: find a k (possibly in terms of c) that makes the inequality hold.

Choose c arbitrarily. How large does n have to be so that $10n^2 \le c\, n^3$?

$$10/c \le n$$

This inequality holds when $n > 10/c$.

So, $k = 10/c$.

Growth of functions - other estimates

For functions f and g, if $f = o(g)$ then $g = \omega(f)$

"g is little-omega of f"

What if $f = o(g)$ and $f = \omega(g)$?

Techniques for solving recurrence relations

No general, automatic procedure for solving is known.

There are methods for solving specific forms of recurrences

- Generate a guess
  - prove the guess is true by induction
- Unrolling the recursion
- Drawing a recursion tree
- Approximate solutions (e.g. $T(n) = O(f(n))$)

- We may or may not cover these:
  - Characteristic equation method
  - Master Theorem
  - Generating functions

Solving recurrence relations

One of the first and most useful recurrences: Compound Interest

Starting with a deposit of \$10,000 with a yield of 11% a year (!)
How much do we have after N years

$T(0) = 10000$
$T(n) = T(n-1) + 0.11\ T(n-1)$

Technique: UNROLLING THE RECURRENCE

$T(n) = 1.11\ T(n-1) =$
$= 1.11\ (1.11\ (T(n-2))$
$= 1.11\ (1.11\ (1.11\ T(n-3))) = 1.11^i T(n-i)$

To get to the base case we let i=n: $T(n) = 1.11^n T(0) = = 1.11^n (10{,}000)$
For n=30 you get \$228,922.97

Solving recurrence relations

T(n) = number of pieces of pizza as a function of the number of cuts

$T(n)=T(n-1)+n$
$T(0)=1$

Technique: UNROLLING THE RECURRENCE

$T(n) = T(n-1)+n$
$T(n) = T(n-2)+(n-1)+n$
$T(n) = T(n-3)+(n-2)+(n-1)+n$

$T(n) = T(n-i)+(n-i+1)+\ldots+n$
To get to the base case we let i=n: $T(n) = T(n-n)+(n-n+1)+\ldots+n$

$$T(n) = T(0)+1+2+\ldots+n = 1+\sum_{i=1}^{n} i$$

$$= \boxed{1+\frac{n(n+1)}{2}}$$

Solving recurrence relations

$T(n)=T(n-1) + 3n$ from this we get $T(n-1) = T(n-2) + 3(n-1)$

$T(0) = 1$

Technique: UNROLLING THE RECURRENCE

$T(n) = T(n-1) + 3n$ substitute

$$T(n) = T(n-2) + 3(n-1) + 3n$$

$$T(n) = T(n-i) + 3(n-i+1) + \dots + 3n$$

$$T(n) = T(0) + 3(1) + 3(2) + \dots + 3n$$

$$= 1 + 3\sum_{i=1}^{n} i = 1 + 3\left(\frac{n(n+1)}{2}\right)$$