Induction
More Examples

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In this lecture, we see more examples of mathematical induction (section 4.1 of Rosen).

1 Recap

A simple proof by induction has the following outline:

> Proof: We will show $P(n)$ is true for all n, using induction on n.
>
> **Base**: We need to show that $P(1)$ is true.
>
> **Induction**: Suppose that $P(k)$ is true, for some integer k. We need to show that $P(k+1)$ is true.

In constructing an induction proof, you've got two tasks. First, you need to set up this outline for your problem. This includes identifying a suitable proposition P and a suitable integer variable n.

Your second task is to fill in the middle part of the induction step. That is, you must figure out how to relate a solution for a larger problem $P(k+1)$ to a solution for a small problem $P(k)$. Most students want to do this by starting with the small problem and adding something to it. For more complex situations, it's usually better to start with the larger problem and try to find an instance of the smaller problem inside it.

2 Finishing up our example

Last Friday, we were trying to prove the following claim:

Claim 1 *For any positive integer n, $\Sigma_{i=1}^{n} i = \frac{n(n+1)}{2}$.*

Let's finish up the details of this example.

> Proof: We will show that $\Sigma_{i=1}^{n} i = \frac{n(n+1)}{2}$ for any integer n, using induction on n.
>
> **Base**: We need to show that the formula holds for $n = 1$. $\Sigma_{i=1}^{1} i = 1$. And also $\frac{1 \cdot 2}{2} = 1$. So the two are equal for $n = 1$.
>
> **Induction**: Suppose that $\Sigma_{i=1}^{k} i = \frac{k(k+1)}{2}$ for some positive integer k. We need to show that $\Sigma_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.
>
> By the definition of summation notation, $\Sigma_{i=1}^{k+1} i = (\Sigma_{i=1}^{k} i) + (k + 1)$
>
> Substituting in the formula from our inductive hypothesis, we get that $(\Sigma_{i=1}^{k} i) + (k + 1) = (\frac{k(k+1)}{2}) + (k + 1)$.
>
> But $(\frac{k(k+1)}{2}) + (k + 1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+2)(k+1)}{2} = \frac{(k+1)(k+2)}{2}$.
>
> So, combining these equations, we get that $\Sigma_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$ which is what we needed to show.

3 Another example

Let's use induction to prove the following claim:

Claim 2 *For every positive integer $n \geq 4$, $2^n < n!$.*

Remember that $n!$ ("n factorial") is $1 \cdot 2 \cdot 3 \cdot 4 \ldots n$. E.g. $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

First, as usual, try some specific integers and verify that the claim is true. Since the claim specifies $n \geq 4$, it's worth checking that 4 does work but the smaller integers don't.

In this claim, the proposition $P(n)$ is $2^n < n!$. So an outline of our inductive proof looks like:

> Proof: Suppose that n is an integer and $n \geq 4$. We'll prove that $2^n < n!$ using induction on n.
>
> Base: $n = 4$. [show that the formula works for $n = 4$]
>
> Induction: Suppose that the claim holds for $n = k$. That is, suppose that we have an integer $k \geq 4$ such that $2^k < k!$. We need to show that $2^{k+1} < (k+1)!$.

Notice that our base case is for $n = 4$ because the claim was specified to hold only for integers ≥ 4.

Fleshing out the details of the algebra, we get the following full proof. When working with inequalities, it's especially important to write down your assumptions and what you want to conclude with. You can then work from both ends to fill in the gap in the middle of the proof.

> Proof: Suppose that n is an integer and $n \geq 4$. We'll prove that $2^n < n!$ using induction on n.
>
> Base: $n = 4$. In this case $2^n = 2^4 = 16$. Also $n! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$. Since $16 < 24$, the formula holds for $n = 4$.
>
> Induction: Suppose that the claim holds for $n = k$. That is, suppose that we have an integer $k \geq 4$ such that $2^k < k!$. We need to show that $2^{k+1} < (k+1)!$.
>
> $2^{k+1} = 2 \cdot 2^k$. By the inductive hypothesis, $2^k < k!$, so $2 \cdot 2^k < 2 \cdot k!$. Since $k \geq 4$, $2 < k+1$. So $2 \cdot k! < (k+1) \cdot k! = (k+1)!$.
>
> Putting these equations together, we find that $2^{k+1} < (k+1)!$, which is what we needed to show.

4 Some comments about style

Notice that the start of the proof tells you which variable in your formula (n in this case) is the induction variable. In this formula, the choice of induction variable is fairly obvious. But sometimes there's more than one integer floating around that might make a plausible choice for the induction variable. It's good style to always mention that you are doing a proof by induction and say what your induction variable is.

It's also good style to label your base and inductive steps.

Notice that the proof of the base case is very short. In fact, I've written about about twice as long as you'd normally see it. Almost all the time, the base case is trivial to prove and fairly obvious to both you and your reader. Often this step contains only some worked algebra and a check mark at the end. The only important thing is that you do actually check the base case: omitting it entirely is a serious flaw.

The important part of the inductive step is ensuring that you assume $P(k)$ and use it to show $P(k+1)$. At the start, you must spell out your inductive hypothesis, i.e. what $P(k)$ is for your claim. Make sure that you use this information in your argument that $P(k+1)$ holds. If you don't, it's not an inductive proof and it's very likely that your proof is buggy.

At the start of the inductive step, it's also a good idea to say what you need to show, i.e. quote what $P(k+1)$ is.

These "style" issues are optional in theory, but actually critical for beginners writing inductive proofs. You will lose points if your proof isn't clear and easy to read. Following these style points (e.g. labelling your base and inductive steps) is a good way to ensure that it is, and that the logic of your proof is correct.

5 Another example

The previous examples applied induction to an algebraic formula. We can also apply induction to other sorts of statements, as long as they involve a suitable integer n.

Claim 3 *For any positive integer n, $n^3 - n$ is divisible by 3.*

In this case, $P(n)$ is "$n^3 - n$ is divisible by 3."

> Proof: By induction on n.
>
> Base: Let $n = 1$. Then $n^3 - n = 1^3 - 1 = 0$ which is divisible by 3.
>
> Induction: Suppose that $k^3 - k$ is divisible by 3, for some positive integer k. We need to show that $(k+1)^3 - (k+1)$ is divisible by 3.
>
> $(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1) = (k^3 - k) + 3(k^2 + k)$
>
> From the inductive hypothesis, $(k^3 - k)$ is divisible by 3. And $3(k^2 + k)$ is divisible by 3 since $(k^2 + k)$ is an integer. So their sum is divisible by 3. That is $(k+1)^3 - (k+1)$ is divisible by 3. $\square$

6 Induction on the size of sets

Now, let's consider a fact about sets which we've used already but never properly proved:

Claim 4 *For any finite set S containing n elements, S has 2^n subsets.*

The objects involved in this claim are sets. To apply induction to facts that aren't about the integers, we need to find a way to use the integers to organize our objects. In this case, we'll organize our sets by their cardinality.

The proposition $P(n)$ for our induction is then "For any set S containing n elements, S has 2^n subsets." Notice that each $P(k)$ is a claim about a whole family of sets, e.g. $P(1)$ is a claim about $\{37\}$, $\{\text{fred}\}$, $\{-31.7\}$, and so forth.

> Proof: We'll prove this for all sets S, by induction on the cardinality of the set.

Base: Suppose that S is a set that contain no elements. Then S is the empty set, which has one subset, i.e. itself. Putting zero into our formula, we get $2^0 = 1$ which is correct.

Induction: Suppose that our claim is true for all sets of k elements, where k is some non-negative integer. We need to show that it is true for all sets of $k + 1$ elements.

Suppose that S is a set containing $k + 1$ elements. Since k is non-negative, $k + 1 \geq 1$, so S must contain at least one element. Let's pick a random element a in S. Let $T = S - \{a\}$.

If B is a subset of S, either B contains a or B doesn't contain a. The subsets of S not containing a are exactly the subsets of T. The subsets of S containing a are exactly the subsets of T, with a added to each one. So S has twice as many subsets as T.

By the induction hypothesis, T has 2^k subsets. So S has $2 \cdot 2^k = 2^{k+1}$ subsets, which is what we needed to show.

Notice that, in the inductive step, we need to show that our claim is true for **all** sets of $k + 1$ elements. Because we are proving a universal statement, we need to pick a representative element of the right type. This is the set S that we choose in the second paragraph of the inductive step.

7 A claim with more than one variable

Here's another useful example that I didn't get to in lecture (and don't plan to do later in the week). Consider this claim:

Claim 5 *For any non-negative integer m and any non-negative real number x, $(1 + x)^m \geq 1 + mx$.*

This claim contains two variables, so it's important to be clear about which is the induction variable. In this case, only m will work because it's the only integer. You can't do induction on real numbers.

So $P(m)$ is "for any non-negative real number x, $(1 + x)^m \geq 1 + mx$."

Proof: by induction on m.

Base: $m = 0$. Then $(1 + x)^m = (1 + x)^0 = 1 = 1 + 0x = 1 + mx$.

Induction: Suppose that there is a non-negative integer k, such that $(1 + x)^k \geq 1 + kx$ for any non-negative real number x. We need to show that $(1 + x)^{k+1} \geq 1 + (k + 1)x$, for any for any non-negative real number x.

$(1 + x)^{k+1} = (1 + x)^k \cdot (1 + x)$. By the induction hypothesis, $(1 + x)^k \geq 1 + kx$. So we have:

$$(1+x)^k \cdot (1+x) \geq (1+kx) \cdot (1+x) = 1+kx+x+kx^2 = 1+(1+k)x+kx^2$$

Since $x^2 \geq 0$ and k was specified to be non-negative, $1 + (1 + k)x + kx^2 \geq 1 + (1 + k)x$. So $(1 + x)^{k+1} \geq 1 + (k + 1)x$, which is what we needed to show.