CS 173: Discrete Structures

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Agenda

- Midterm on Wednesday
  - Bring your IDs
  - Last semester's first mid-term:

  http://www.cs.uiuc.edu/class/fa08/cs173/Exams/midterm1-answers.pdf

  NOTE: TOPICS NOT NECESSARILY THE SAME!

- Today: Functions

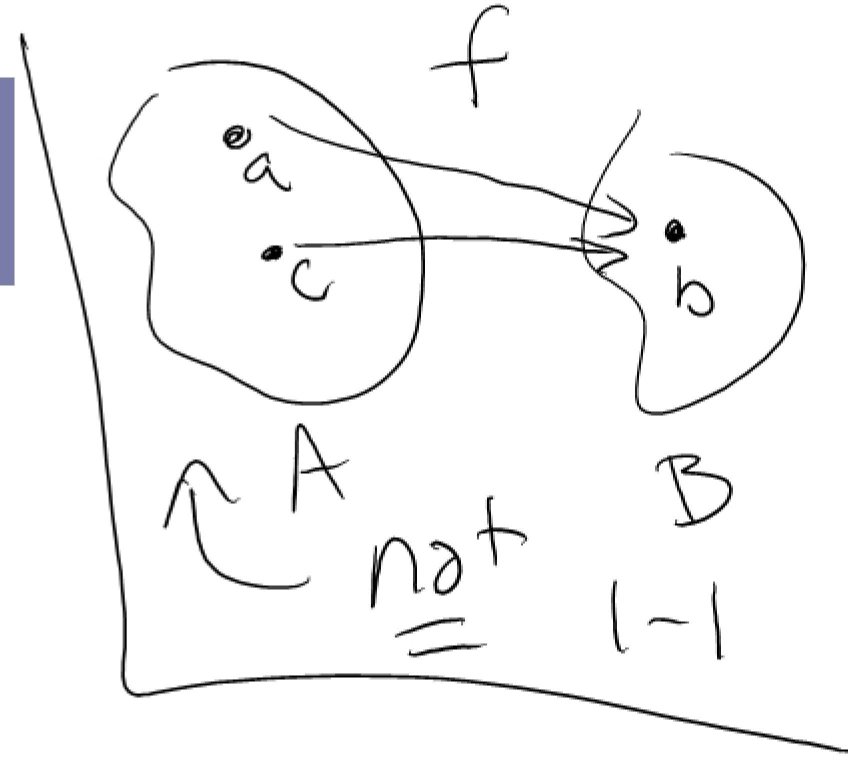
$$\sum_{i=?}^{n} \text{[illegible]} = \boxed{f(n)}$$

"closed form"

Functions for the mid-term

- Know the basic notation $f{:}A \rightarrow B$
- Know the definition of a function
- Know the meaning of the terms *domain* and *co-domain*
- Know when a function is *one-to-one* (injective)
- Know when a function is *onto* (surjective)
- Identify if a function is one-to-one and/or onto.

Functions - injection



A function f: $A \to B$ is one-to-one (injective, an injection) if $\forall a,b,c,\ (f(a) = b \wedge f(c) = b) \to a = c$

$Z^+ = \{1, 2, \ldots\}$

$R^+ \ \{x \in \mathbb{R} : x > 0\}$

Are these injective:

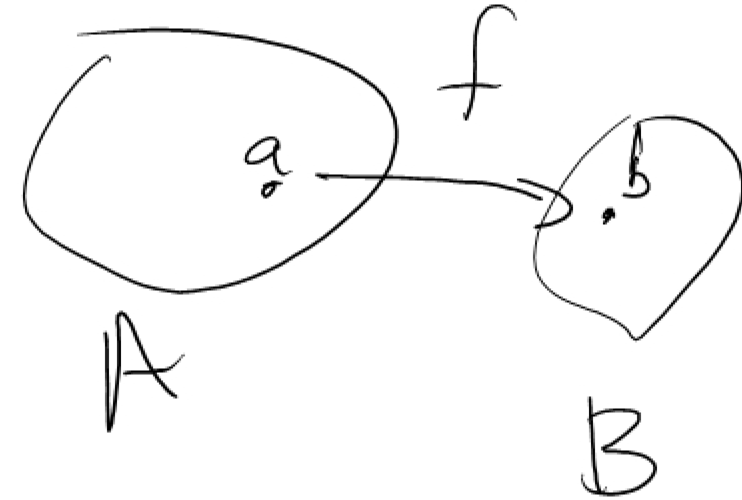
1) $f: R^+ \to R^+$ $\quad f(x) = 1/(x+1)$ yes

2) $f: Q \to Q^{\geq 0}$ $\quad f(x) = |3x+1|$ no

$f(-2) = 5$ not 1-1

$f(4/3) = 5$

Functions - surjection



A function f: $A \to B$ is onto (surjective, a surjection) if $\forall b \in B, \exists a \in A\ f(a) = b$

Are these surjective:

1) f: $\mathbb{N} \to \mathbb{N}$ $f(x) = 3x+1$ no

if $x \in N$ $2 \neq 3x+1$

2) f: $\mathbb{R} \to \mathbb{N}$ $f(x) = 3x+1$ yes

for $y \in N$, $y = 3x+1$ if follows

$x = \frac{y-1}{3}$ $x \in R$

Functions - bijection

A function $f: A \rightarrow B$ is bijective if it is one-to-one and onto.

Are these bijective:

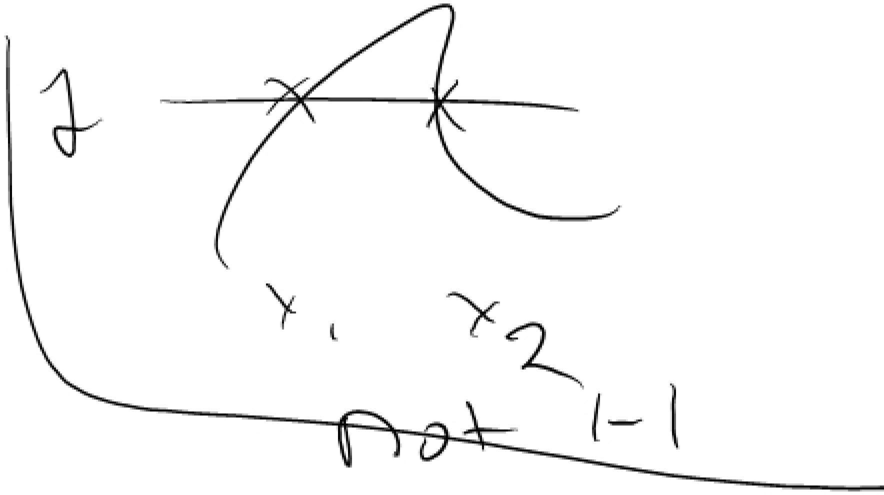
1) $f: Z \rightarrow Z$ $f(x) = 2x+3$ no not onto

2) $f: Z \rightarrow N$ $f(x) = -2x$ if $x <= 0$ $= \{0, 2, 4, \ldots\}$

$2x-1$ if $x > 0$ $= \{1, 3, 5, \ldots\}$

yes

Functions - examples



Changing the domain and co-domain can change the properties of the function

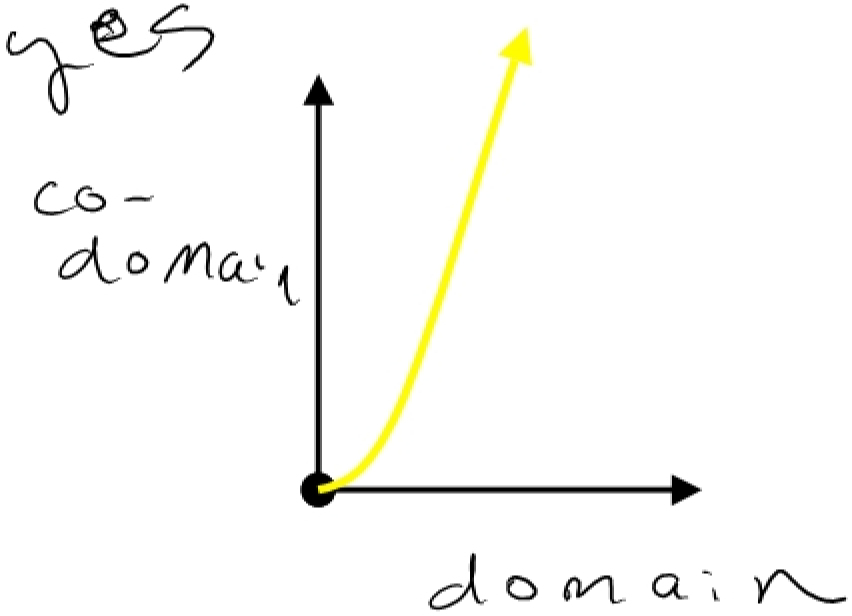
Suppose $f: R^+ \to R^+$, $f(x) = x^2$.

Is f one-to-one? yes

Is f onto? yes

Is f bijective?

yes



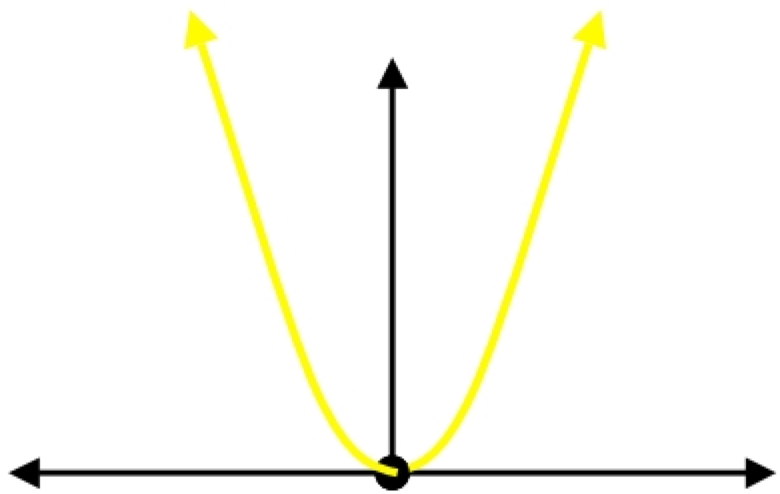
Functions - examples

Suppose $f: R \to R^+$, $f(x) = x^2$.

Is f one-to-one? no

Is f onto? yes

Is f bijective? no



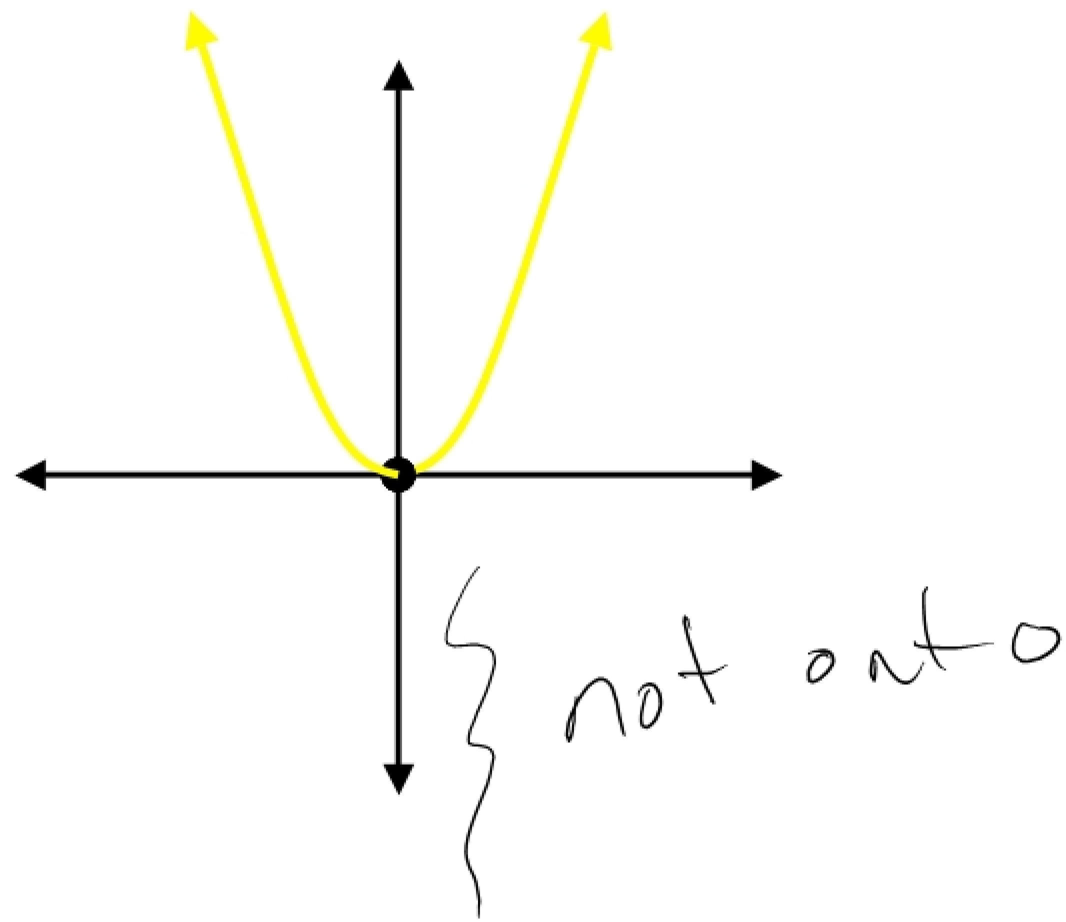
Functions - examples

Suppose $f: R \rightarrow R$, $f(x) = x^2$.

Is f one-to-one? no

Is f onto? no

Is f bijective? no



Proving f is one-to-one

Let f : Z $\rightarrow$ Z be defined by f(x) = 3x + 7

Prove f is one-to-one.

How do we start? Think of the def. of one-to-one.

We need to show that for all x,y $\in$ Z

If f(x) =f(y) then x =y

Proving f is one-to-one

Let $f : Z \to Z$ be defined by $f(x) = 3x + 7$

Prove f is one-to-one.

If $f(x) = f(y) \to x = y$ $\forall x, y \in Z$ then
$f(x)$ will be 1-1

Assume $f(x) = f(y)$

$$3x + 7 = 3y + 7$$

$$3x = 3y$$

$$x = y$$

So f is one to one

Proving g is onto

Let $g : Z \to Z$ be defined by $g(x) = \cancel{g} + 3$

x

Prove g is onto.

How do we start? Think of the def. of onto

We need to show that for all $y \in Z$

There is some $x \in Z$ such that $g(x) = y$

Let $y \in \mathbb{Z}$

Let $\boxed{x = y - 3}$

It follows $x \in \mathbb{Z}$

And $g(x) =$

$(y-3)+3$

$= y$

So $\exists x \in \mathbb{Z}$ s.t.

$g(x) = y$

$\forall y \in \mathbb{Z}$

Inverses

Functions $f: A \to B$ and $g: B \to A$ are **inverses** if $f(a) = b$ iff $g(b)=a$ for all $a \in A$ and $b \in B$

The inverse of f is denoted $\boxed{f^{-1}}$

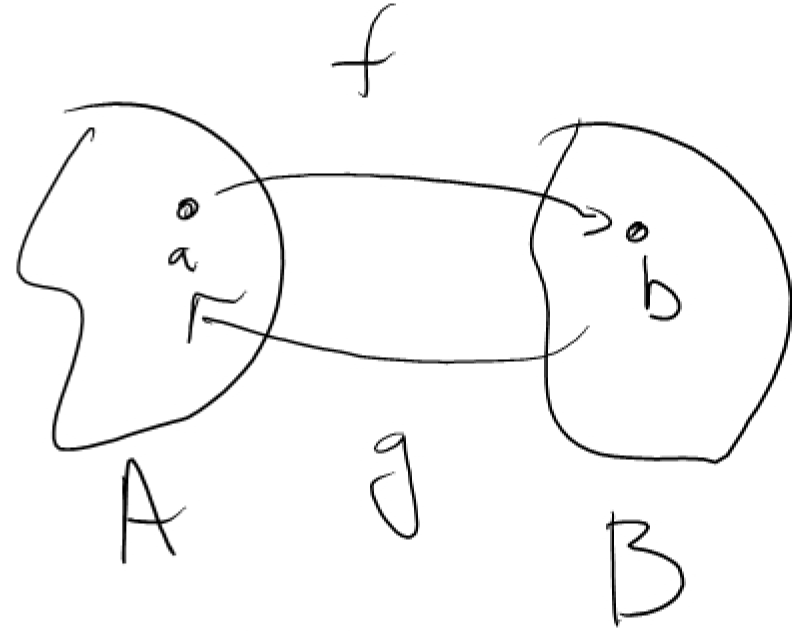
Do all functions have inverses?

no

What conditions are required for f^{-1} to exist?

f is one to one

f is onto



$f(a) = b$

$g(b) = a$

Inverses

Functions f: $A \rightarrow B$ and g: $B \rightarrow A$ are **inverses** if $f(a) = b$ iff $g(b)=a$ for all $a \in A$ and $b \in B$

$$g(y) = \frac{5}{2}(y+2)$$

What is the inverse of the function $f: \mathbb{Q} \rightarrow \mathbb{Q}$
$f(x) = (2/5)x - 2$?

$$\left(\frac{2}{5}\right)x - 2 = y$$

scratch

$$\frac{2}{5}x = y+2$$

$$x = \frac{5}{2}(y+2)$$

What is the inverse of the function $f: \mathbb{Q} \rightarrow \mathbb{Q}$
$f(x) = 2x+8$?

$$2x + 8 = y$$

$$\boxed{g(y) = \left(\frac{1}{2}\right)y - 4}$$

Proofs about inverses

if f: Q → Q $f(x) = 3x+9$

and g: Q→Q $g(y) = y/3-3$, prove f and g are inverses

(1) if $f(x) = y$ then $g(y) = x$

similar to

(2) if $\underbrace{g(y) = x}$ then $f(x) = y$

$g(y) = x$

$\frac{y}{3} - 3 = x$

$y = 3(x+3)$

$y = \underbrace{3x+9}$

so $f(x) = y$

Functions - composition

Let $f: A \rightarrow B$, and $g: B \rightarrow C$ be functions.

Then the composition of f and g is:

$(g \circ f)(x) = g(f(x))$

Example: Let $g(x) = x^2$ for $x \in \mathbb{Z}$

Let $f(x) = 3x+4$ for $x \in \mathbb{Z}$

What is $(g \circ f)(x)$? $= g(3x+4) = (3x+4)^2$

$\boxed{g \circ f(x) = g(f(x))}$

$A \rightarrow C$

Proofs about composition

Let $f:A \to B$, and $g:B \to C$ be functions.

Prove that if f and g are injective that $(g \circ f)(x)$ is injective.

What do we need to show?

Show if $g \circ f(x) = g \circ f(y)$ then $x = y$

$\forall x, y \in A$

Proofs about composition

$g \circ f(x)$

$= \boxed{g(f(x))}$

Let $f{:}A \to B$, and $g{:}B \to C$ be functions.

Prove that if f and g are injective that $(g \circ f)(x)$ is injective.

$f \circ g(x) =$

$f(g)$

Suppose $g(f(x)) = g(f(y))$ for $x, y \in A$

Since g is injective $f(x) = f(y)$ since g is 1-1

Since f is injective $x = y$

since f is 1-1

We've shown $g(f(x)) = g(f(y)) \rightarrow x = y \ \forall x, y \in A$

So gof is 1-1