Set theory proof example

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16 February 2009

We saw several set theory proofs in class (see slides). Here's one of the more complex ones written up neatly, as a model to use in writing your own proofs.

1 A set theory proof with cartesian products

If we want to show that a set A is a subset of a set B, a standard proof outline involves picking a random element x from A and then showing that x must be in B. For example, consider the claim:

Claim 1 *For any sets A, B, C, and D, if $A \subseteq B$ and $C \subseteq D$, then $A \times C \subseteq B \times D$.*

To start, let's pick some sets (assuming nothing special about them) and assume that all the hypotheses of the claim are true.

> Proof draft 1: Suppose that A, B, C, and D such that if $A \subseteq B$ and $C \subseteq D$.
>
> We need to show that $A \times C \subseteq B \times D$.

In order to move forwards, we need to realize that $A \times C \subseteq B \times D$ can be translated into an if/then statement. If $w \in A \times C$ then $w \in B \times D$. So proving this means that we do another step of assuming the hypothesis and saying we need to prove the conclusion.

> Proof draft 2: Suppose that A, B, C, and D such that if $A \subseteq B$ and $C \subseteq D$.
>
> We need to show that $A \times C \subseteq B \times D$. So suppose that w is an element of $A \times C$. We need to show that $w \in B \times D$.

To show this, we need to have a clear mental picture of the objects we're manipulating. What type of thing is w? It's a member of $A \times C$, so it must be an ordered pair. To manipulate an ordered pair, we often have to name its two components, e.g. (x, y).

> Proof: Suppose that A, B, C, and D such that if $A \subseteq B$ and $C \subseteq D$.
>
> We need to show that $A \times C \subseteq B \times D$. So suppose that w is an element of $A \times C$. We need to show that $w \in B \times D$.
>
> Since w is an element of $A \times C$, we can write it as $w = (x, y)$ where $x \in A$ and $y \in C$ (by the definition of Cartesian product).
>
> Since $x \in A$ and $A \subseteq B$, $x \in B$. Since $y \in C$ and $C \subseteq D$, $y \in D$. So, using the definition of Cartesian product again, $(x, y) \in B \times D$. That is, $w \in B \times D$.
>
> We've shown that any element of $A \times C$ is also an element of $B \times D$. By the definition of subset, this means that $A \times C \subseteq B \times D$.