Crash Introduction to Quantifiers

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This lecture finishes the quick introduction to quantifiers, which follows part of section 1.3 of Rosen. We'll also see some examples of direct proof, from section 1.6 of Rosen. Don't worry about reading either section right now. We'll do that when we're closer to covering their full content.

1   Announcements

Turn in your homework: please put it in the folder for your section on the back table.

Announcement from Fernanda Mendes about the UIUC WCS ("Women in Computer Science") mentoring program. Match you up with corporate mentors. Open to everyone (not just women). Details on their web site.

2   Recap

Recall from last lecture that a predicate (e.g. $x^2 \geq 0$) is a statement that becomes true or false when you substitute particular values for its variables (e.g. $3^2 \geq 0$). Predicates are used to make general claims about whole sets of values, such as "For every integer x, $x^2 \geq 0$." Notice that we carefully specified the type of x, i.e. what types of values (integers in this case) that we allow to be substituted for the variable x. The set of values that can be substituted in for a variable is called its *domain* or its *replacement set*.

3   Quantifiers

The operation "for all x" is formally known as a *universal quantifier*.

The other heavily-used quantifier in formal mathematics is the *existential quantifier*, written "there exists an x" For example, "there exists an integer x such that $5 < x < 100$." This means that at least one integer x (and possibly a whole bunch of integers) satisfies the equation.

Notice that the existential quantifer is followed by "such that" in fluent mathematical English, which isn't quite parallel to what you see with the universal quantifier. This isn't for any good reason. It's just how mathematical English happens to have evolved, and you should do likewise so that your written mathematics looks professional. "Such that" is sometimes abbreviated "s.t."

The general idea of a quantifier is that it expresses express how many of the values in the domain make the claim true. Normal English has a wide range of quantifiers which tell you roughly how many of the values work, e.g. "some", "a couple", "a few", "many", "most." Mathematics largely uses just the two we've seen, though you occasionally see a third quantifier "there exists a unique x." This means that one and only one x satisfies the predicate. For example, "there is a unique integer x such that $x^2 = 0$." Mathematicians use the adjective "unique" to mean that there's only one such object (similar to the normal usage but not quite the same).

In a statement like $\forall x \in \mathbb{N}, x \le 2x$, the quantifier $\forall$ is said to "bind" its variable x. Similarly, summations and integrals bind their dummy variables, e.g. the i in $\sum_{i=0}^{n} \frac{1}{i}$. Variables that aren't bound are called "free." Statements containing free variables don't have a defined truth value.

4   Shorthand notation

The universal quantifier has the shorthand notation $\forall$. For example, $\forall x \in \mathbb{N}, x \le 2x$. The existential quantifier is written $\exists$, e.g. $\exists y \in \mathbb{R} y = \sqrt{2}$. Notice that we don't write "such that" when the quantifier is in shorthand. The unique existence quantifier is written $\exists!$ as in $\exists! x \in \mathbb{R}, x^2 = 0$.

There's several conventions about inserting commas after the quantifier and/or parentheses around the following predicate. The style in these notes and the style in Rosen are both ok to copy from.

If you want to state a claim about two numbers, you can use two quantifiers as in:

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y \ge x$$

This is usually abbreviated to

$$\forall x, y \in \mathbb{R}, x + y \geq x$$

This means "for all real numbers x and y, $x + y \geq x$" (which isn't true).

When manipulating 2D points, you have several options. You can write something like $\forall x, y \in \mathbb{Z}$ ("for any integers x and y") and then later refer to the ordered pair (x, y). Or, people also write $\forall (x, y) \in \mathbb{Z}^2$ ("For any point (x, y) with integer coordinates"). Or you can write something like $\forall p \in \mathbb{Z}^2$ ("for any 2D point p with integer coordinates"). Then, when you later need to access its two coordinates, you can assert (based on our knowledge of 2D geometry) that p has the form (x, y), where x and y are integers. All three approaches work ok.

5   Negating statements with quantifiers

In general, a statement of the form "for all x in A, $P(x)$" is false exactly when there is some value x in A such that $P(x)$ is false. In other words, when "there exists x in A such that $P(x)$ is not true". In shorthand notation:

$$\neg(\forall x, P(x)) \equiv \exists x, \neg P(x)$$

Similarly,

$$\neg(\exists x, P(x)) \equiv \forall x, \neg P(x)$$

So this is a bit like the de Morgan's laws: when you move the negation across the operator, you change it to the other similar operator.

Last class, we saw how to move negation operators from the outside to the inside of expressions involving $\wedge$, $\vee$, and the other propositional operators. Together with these two new rules to handle quantifiers, we now have a mechanical procedure for working out the negation of any random statement in predicate logic.

So if we have something like

$$\forall x, P(x) \rightarrow (Q(x) \wedge R(x))$$

It's negation is

$$\neg(\forall x, P(x) \rightarrow (Q(x) \wedge R(x))) \equiv \exists x, \neg(P(x) \rightarrow (Q(x) \wedge R(x)))$$
$$\equiv \exists x, P(x) \wedge \neg(Q(x) \wedge R(x)))$$
$$\equiv \exists x, P(x) \wedge (\neg Q(x) \vee \neg R(x))$$

6 Proving a universal statement

Now, let's consider how to prove a claim like

> For every rational number q, $2q$ is rational.

First, we need to define what we mean by "rational".

> A real number r is *rational* if there are integers m and n, $n \neq 0$, such that $r = \frac{m}{n}$.

The simplest technique for proving a claim of the form $\forall x \in A, P(x)$ is to pick some representative value for x.[1] Think about sticking your hand into the set A with your eyes closed and pulling out some random element. You use the fact that x is an element of A to show that $P(x)$ is true. Here's what it looks like for our example:

> Proof: Let q be any rational number. From the definition of "rational," we know that $q = \frac{m}{n}$ where m and n are integers and n is not zero. So $2q = 2\frac{m}{n} = \frac{2m}{n}$. Since m is an integer, so is $2m$. So $2q$ is also the ratio of two integers and, therefore, $2q$ is rational.

At the start of the proof, notice that we expanded the word "rational" into what its definition said. At the end of the proof, we went the other way: noticed that something had the form required by the definition and then asserted that it must be a rational.

Notice also that we spelled out the definition of "rational" but we just freely used facts from high school algebra as if they were obvious. In general, when writing proofs, you and your reader come to some agreement about what parts of math will be considered familiar and obvious, and which require explicit discussion. For this course, basic facts from high school math courses will be considered obvious.

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[1] The formal name for this is "universal instantiation."

7 A couple notational issues

WARNING!! Abuse of notation. Notice that our definitions said "if". If you take them literally as read, that means you can only use them in one direction. This isn't what's meant. Definitions are always intended to work in both directions, i.e. I meant to say "if and only if." This little misuse of "if" in definitions is very, very common.

Also notice that "iff" is shorthand for "if and only if", which is the same thing as $\leftrightarrow$.

8 Proving existential statements

Here's an existential claim:

Claim 1 *There is an integer k such that $k^2 = 0$.*

An existential claim such as the following asserts the existence of an object with some set of properties. So it's enough to exhibit some specific concrete object, of our choosing, with the required properties. So our proof can be very simple:

> Proof: Zero is such an integer. So the statement is true.

We could spell out a bit more detail, but it's really not necessary. Proofs of existential claims are often very short, though there are exceptions.

Notice one difference from our previous proofs. When we pick a value to instantiate a universally quantified variable, we have no control over exactly what the value is. We have to base our reasoning just on what set it belongs to. But when we are proving an existential claim, we get to pick our own favorite choice of concrete value, in this case zero.