Propositional Equivalences, Intro to Predicates

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In this lecture, we will see how to show that two propositional formulas are logically equivalent (Section 1.2 of Rosen). We will also see some of the material on predicates and quantifiers from section 1.3 of Rosen.

1 Announcements

One goal of Homework 0 is to help everyone to remember, or quickly learn, material they are supposed to know from previous classes. So different people may have trouble with different problems. Problem 6 may be mysterious to students with no programming background. So it's a test of how fast you can get up to speed enough to survive later in this course. Have a look at Appendix 3, come to office hours, ask older students for advice, and see how much of it you can figure out.

2 Recap

Last class, we saw the meanings of several logical operators, along with shorthand notation: $\vee$ (or), $\wedge$ (and), $\rightarrow$ (implies), $\leftrightarrow$ (implies in both directions), $\neg$ (not), $\oplus$ (exclusive or).

3 Logical Equivalence

Two compound propositions p and q are *logically equivalent* if they are true for exactly the same input values. The shorthand notation for this is $p \equiv q$. One way to establish logical equivalence is with a truth table.

For example, last class we saw that implication has the truth table:

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

A frequently useful fact is that $p \rightarrow q$ is logically equivalent to $\neg p \vee q$. To show this, built the truth table for $\neg p \vee q$. and compare the output values to those for $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

Two very well-known equivalences are *De Morgan's Laws*. These state that $\neg(p \wedge q)$ is equivalent to $\neg p \vee \neg q$. and that $\neg(p \vee q)$ is equivalent to $\neg p \wedge \neg q$. Similar rules in other domains (e.g. set theory) are also called De Morgan's Laws. They are especially helpful, because they tell you how to simplify the negation of a complex statement involving "and" and "or".

We can show this easily with another truth table:

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Truth tables are a nice way to show equivalence for compound propositions which use only 2-3 variables. For more variables, they are cumbersome because analyzing the effects of k input variables requires 2^k rows.

4 Some useful logical equivalences

Rosen pp. 24-25 (reproduced in handout on-line) lists a number of useful logical equivalences, which you could establish with truth tables (but probably don't want to). Browse through them at your leisure. Some closely (and for good mathematical reasons) resemble rules from algebra and/or set theory, sometimes sharing the same name.

For example, one of the two distributive laws states that

$$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$$

which is similar to the familiar rule

$$a \times (b + c) = (a \times b) + (a \times c)$$

except that the logic expression requires parentheses and also works with $\lor$ and $\land$ interchanged.

5 Negating propositions

An important use of logical equivalences is to help you correctly state the negation of a complex proposition, i.e. what it means for the complex proposition not to be true. Having clear mechanical rules for negation is important when working with concepts that are new to you, so you have only limited intuitions about what is correct.

For example, suppose we have a claim like "If M is regular, then M is paracompact or M is not Lindelöf." I'm sure you have no idea whether this is even true, because it comes from a math class you are almost certain not to have taken. However, you can figure out its negation.

First, let's convert the claim into shorthand so we can see its structure. Let r be "M is regular", p be "M is paracompact", and l be "M is Lindelöf." Then the claim would be $r \rightarrow (p \lor \neg l)$.

The negation of $r \rightarrow (p \lor \neg l)$ is $\neg(r \rightarrow (p \lor \neg l))$. However, to do anything useful with this negated expression, we need to manipulate it into an equivalent expression that has the negations on the individual propositions.

The key equivalences used in doing this manipulation are:

- $\neg(\neg p) \equiv p$
- $\neg(p \land q) \equiv \neg p \lor \neg q$
- $\neg(p \lor q) \equiv \neg p \land \neg q$
- $p \rightarrow q \equiv \neg p \lor q$.

So we have

$$\neg(r \rightarrow (p \vee \neg l)) \equiv r \wedge \neg(p \vee \neg l) \equiv r \wedge \neg p \wedge \neg\neg l \equiv r \wedge \neg p \wedge l$$

So the negation of our original claim is "M is regular and M is not paracompact and M is Lindelöf." Knowing the mechanical rules helps you handle situations where your logical intuitions aren't fully up to the task of just seeing instinctively what the negation should look like.

6 Some useful terminology

T and F are special constant propositions with no variables that are, respectively, always true and always false.

A compound proposition p that is true for all combinations of input values is called a *tautology*. Or, equivalently, p is logically equivalent to T. A compound proposition p that is false for all combinations of input values (aka logically equivalent to F) is called a *contradiction*. So T and F are especially simple examples of a tautology and an contradiction.

Two compound propositions p and q are logically equivalent exactly when $p \leftrightarrow q$ is a tautology.

7 Establishing new logical equivalences

These established logical equivalences can be used to show that other, novel or more complex, equivalences also hold. You are already familiar with the technique used to do these proofs: long chain of equations.

For example, to show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$, we show the following sequence of equations.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && (1)\\
&\equiv \neg p \wedge (\neg(\neg p) \vee \neg q && (2)\\
&\equiv \neg p \wedge (p \vee \neg q) && (3)\\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && (4)\\
&\equiv (p \wedge \neg p) \vee (\neg p \wedge \neg q) && (5)\\
&\equiv F \vee (\neg p \wedge \neg q) && (6)\\
&\equiv (\neg p \wedge \neg q) \vee F && (7)\\
&\equiv \neg p \wedge \neg q && (8)
\end{aligned}
$$

Steps (1) and (2) use the two De Morgan's laws. Step (3) uses the fact that negations cancel. Step (4) uses a distributive law, and so forth.

For the purposes of the homework problems on this material (Homework 1) please be *very* picky about matching the exact form of the rules in our tables. Pretend you are a computer and therefore very picky and lacking in common sense. As we get into more complex material, we'll stop being so picky about every small step. For example, step (6) uses a commutative law to put the formula in the exact right form to apply the identity law for F. In real life (i.e. a couple homeworks down the road) one would just proceed directly from (5) to (7) without bothering to name the formula justifying the transformation.

8 Predicates and Variables

Propositions are a helpful beginning but too rigid to represent most of the interesting bits of mathematics. To do this, we need predicate logic, which allows variables and predicates that take variables as input. We'll get started with predicate logic now, but delay covering some of the details (e.g. sections 1.3 and 1.4) until they become relevant to the proofs we're looking at.

A predicate is a statement that becomes true or false if you substitute in values for its variables. For example, "$x^2 \geq 10$" or "y is winter hardy." Suppose we call these $P(x)$ and $Q(x)$. Then $Q(x)$ is true if x is "mint" but not if x is "tomato".[1]

[1] A winter hardy plant is a plant that can survive the winter in a cold climate, e.g. here in central Illinois.

If we substitute concrete values for all the variables in a predicate, we're back to having a proposition. That wasn't much use, was it?

The main use of predicates is to make general statements about what happens when you substitute a variety of values for the variables. For example:

P(x) is true for every x

For example, "For every integer x, $x^2 \geq 10$" (false).

Consider "For all x, $2x \geq x$." Is this true or false? This depends on what values x can have. Is x any integer? In that case the claim is false. But if x is supposed to be a natural number, then the claim is true.

In order to decide whether a statement involving quantifiers is true, you need to know what types of values are allowed for each variable. Good style requires that you state the type of the variable explicitly when you introduce it, e.g. "For all natural numbers x, $2x \geq x$." Exceptions involve cases where the type is very, very clear from the context, e.g. when a whole long discussion is all about (say) the integers. If you aren't sure, a redundant type statement is a minor problem whereas a missing type statement is sometimes a big problem.