Math Basics

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This lecture reviews key mathematical concepts, some familiar and some not, that will be helpful in the course. This material can be found in Rosen: end part of section 2.3, section 2.4, and appendix 2.

1 Announcements

Homework 0 should be posted on the web page sometime this morning, due in class next Friday.

Office hours (starting next week) are almost finalized and will appear on the web page probably this afternoon. Feel free to come to anyone's office hour, not just your section leader's.

Proficiency exams will be next Tuesday (except maybe for one person with a difficult schedule). If you plan to take the exam and haven't gotten email about the times and room, bug Margaret by email (mfleck@cs.uiuc.edu).

I haven't responded to everyone who sent me mail about the honors section. Over the next week or so, I'll compare those emails to the registration list and the honors mailing list, to make sure they are all consistent.

2 Overview

We're assuming that you understood precalculus when you took it. So you used to know how to do things like factoring polynomials, solving high school geometry problems, using trigonometric identities. However, you probably can't remember it all cold. Some things will appear on homeworks and we'll expect that you can look them up. For example, one question on Homework

0 expects you to remember a very well-known fact about right triangles. Wikipedia is often a good place to find forgotten identities.

Today we're going to go over some concepts and notation that we'll use a lot during the course, plus a couple concepts that are fairly easy but some of you might not have seen.

3 Some sets

You've all seen sets, though probably a bit informally. We'll get back to some of the formalism in a couple weeks. Meanwhile, here are the names for a few commonly used sets of numbers:

- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the integers.
- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the natural numbers.
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ is the natural numbers.
- $\mathbb{R}$ is the real numbers
- $\mathbb{Q}$ is the rational numbers
- $\mathbb{C}$ is the complex numbers

Notice that, in this class, the natural numbers contains zero. Authors differ as to whether zero is in the natural numbers and there is no clear standard. When you get out of this class, always check which convention an author is following.

Remember that the rational numbers are the set of fractions $\frac{p}{q}$ where q can't be zero and we consider two fractions to be the same number if they are the same when you reduce them to lowest terms.

The real numbers contain all the rationals, plus irrational numbers such as $\sqrt{2}$ and π. (Also e, but we don't use e much in this end of computer science.)

The complex numbers are numbers of the form $a + bi$, where a and b are real numbers and i is the square root of -1. Some of you from ECE will be aware that there are many strange and interesting things that can be done with the complex numbers. We won't be doing that here. We'll just use them for a few examples and expect you to do very basic things, mostly easy consequences of the fact that $i = \sqrt{-1}$.

Notice that $\sqrt{-1}$ is named j over in ECE and physics, because i is current. For this class, please use i, since that's the convention in computer science (and big parts of math).

If we want to say that the variable x is a real, we write $x \in \mathbb{R}$, and similarly for the other sets. In calculus, you don't have to think much about the types of your variables, because they are largely reals. In this class, we handle variables of several different types, so it's important to state the types explicitly.

It is sometimes useful to select the real numbers in some limited range. We use the following notation:

- $[a, b]$ is the set of real numbers from a to b, including a and b
- (a, b) is the set of real numbers from a to b, not including a and b
- $[a, b)$ and $(a, b]$ include only one of the two endpoints.

4 Stay in the set you are using

If someone gives you a problem about integers, it's normally best to do all your computations using integers. For example, don't divide two of the numbers, because that would create a rational number. Similarly, we won't use calculus in this course. Don't use calculus on problems that don't require it.

This warning is for several reasons. First, it's not elegant to deploy a nuclear weapon to kill an ant. An approach directly using the simpler objects often yields a simpler argument.

Second, it's interesting to understand the simpler systems in their own terms, partly because we're deriving facts that are part of what's used to construct the more complex sets (e.g. the reals).

Third, you can get into trouble if you step outside the domain of the problem. For example, functions defined on the integers don't have derivatives, so some fancy footwork is required to apply calculus arguments to them. Similarly, in computer programs, integer operations yield exact answers but reals have to be approximated with floating point. So shifting to reals creates errors in your computation.

There are, of course, exceptions. But I don't think we'll see any of them this term.

5 Pairs of reals

The set of all pairs of reals is written $\mathbb{R}^2$. So it contains pairs like $(-2.3, 4.7)$. In a computer program, we could implement such a pair using an object or struct. But what does the pair represent? It might be

- a point in 2D space
- a complex number
- a rational number (if the second coordinate isn't zero)
- an interval of the real line

So, when you see notation like (x, y), you have to ask yourself what the author intended this object to be.

The intended meaning affects what operations we can do on the pairs. For example, if (x, y) is an interval of the real line, it's a set. So we can write things like $z \in (x, y)$ (z is in the interval (x, y)). That notation would be meaningless if (x, y) is a 2D point or a number.

If the pairs are numbers, we can add them, but the result depends on what they are representing. So $(x, y) + (a, b)$ is $(x + a, y + b)$ for 2D points and complex numbers. But it is $(xb + ya, yb)$ for rationals.

Suppose we want to multiply $(x, y) \times (a, b)$ and get another pair as output. There's no obvious way to do this for 2D points. For rationals, the formula would be (xa, yb), but for complex numbers it's $(xa - yb, ya + xb)$.

Stop back and work out that last one for the non-ECE students, using more familiar notation. $(x + yi)(a + bi)$ is $xa + yai + xbi + byi^2$. But $i^2 = -1$. So this reduces to $(xa - yb) + (ya + xb)i$.

Oddly enough, you can also multiply two intervals of the real line. This carves out a rectangular region of the 2D plane, with sides determined by the two intervals.

6 Exponentials and logs

Suppose that b is any real number. We all know how to take integer powers of b, i,e. b^n is b multiplied by itself n times. It's not so clear how to precisely define b^x, but we've all got faith that it works (e.g. our calculators produce values for it) and it's a smooth function that grows really fast as the input gets bigger and agrees with the integer definition on integer inputs.

Here are some special cases to know

- b^0 is one for any b.
- $b^{0.5}$ is $\sqrt{b}$
- b^{-1} is $\frac{1}{b}$

And some handy rules for manipulating exponents:
$b^x b^y = b^{x+y}$
$(b^x)^y = b^{xy}$
$b^{x^y} \neq b^{xy}$ (This is a common mistake.)

Suppose that $b > 1$. Then we can invert the function b^x, to get the function $\log_b x$ ("logarithm of x to the base b"). Draw the graph. Logarithms appear in computer science as the running times of particularly fast algorithms. They are also used to manipulate numbers that have very wide ranges, e.g. probabilities.

Notice that the log function takes only positive numbers as inputs. In this class, $\log x$ with no explicit base always means $\log_2 x$ because digital computers make such heavy use of base-2. In many other fields, the default base is e, so that various continuous mathematical formulas will work out nicely.

Useful facts about logarithms include:
$\log_b(xy) = \log_b(x) + \log_b y$
$\log_b(x^y) = y \log_b(x)$
Change of base formula: $\log_b x = \log_a x \log_b a$

Notice that the multiplier to change bases is a constant, i.e doesn't depend on x. So it just shifts the curve up and down without really changing its shape. In many computer science analyses, we don't care about constants. So we often write $\log x$ when we just don't care about what the base is.

7 Floor and ceiling

The floor and ceiling functions are heavily used in computer science, though not in many areas of science and engineering. Both functions take a real number x as input and return an integer near x. The floor function returns the largest integer no bigger than x. In other words, it converts x to an integer, rounding down. This can be written floor(x) or, in shorthand, $\lfloor x \rfloor$. If the input to floor is already an integer, it is returned unchanged. Notice that floor rounds downward even for negative numbers. So

$\lfloor 3.75 \rfloor = 3$
$\lfloor 3 \rfloor = 3$
$\lfloor -3.75 \rfloor = -4$

The ceiling function is similar, but rounds upwards. So

$\lceil 3.75 \rceil = 4$
$\lceil 3 \rceil = 3$
$\lceil -3.75 \rceil = -3$

Most programming languages have these two functions, plus a function that rounds to the nearest integer and one that "truncates" i.e. rounds towards zero. Round is often used in statistical programs. Truncate isn't used much in theoretical analyses.

8 Summations

If a_i is some formula that depends on i, then

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

For example

$$\sum_{i=1}^{n} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \ldots + \frac{1}{2^n}$$

Products can be written with a similar notation, e.g.

$$\prod_{k=1}^{n} \frac{1}{k} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \ldots \cdot \frac{1}{n}$$

Certain sums can be re-expressed "in closed form" i.e. without the summation notation. For example:

$$\sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$$

In Calculus, you may have seen the infinite version of this sum, which converges to 1. In this class, we're always dealing with finite sums, not infinite ones.

If you modify the start value, so we start with the zeroth term, we get the following variation on this summation. Always be careful to check where your summation starts.

$$\sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n}$$

Section 2.4 has a table of some of the more useful formulas for summations that have simple closed forms. You don't have to memorize them, but they may be useful in homeworks. We'll see them again when we cover mathematical induction and see how to formally prove that they are correct.

One famous sum that you should memorize is

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Here's one way to convince yourself it's right. Fill in rows on a sheet of graph paper, first one unit, next two units, etc up to n units. Surround this little triangular pattern with a box n units high and $n+1$ units wide. The filled part is exactly half the area of the box, which is $n(n+1)$ units.