CS 173: Discrete Structures

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Office Hour: Wed. 1-2, 2215 SC

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Agenda

- About the Quizzes
  - They'll be returned by mid-week
- Midterm on 2/25 (a week from Wednesday)
- Today: Conclusion of Set Theory
  - Section 2.2
    - Operations on Sets
    - Proving things…

One last word about RNGs

- “The major difference between Nevada casinos and the Montreal Casino, however, is that the Nevada casinos operate 24 hours per day, never turning off their keno games, while the Montreal Casino shuts down each night and reopens again in the morning. Without the clock chip to generate different seeds, each day the Montreal Casino was cycling through the same numbers, beginning at the same starting point! This is what Daniel Corriveau discovered. And this discovery paid him $600,000 in keno winnings.”

Set Theory - Definitions and notation

What is the cardinality of this set: 7

$$\{a,b,c,\{a\},\{a,b,c\},\{c\}, \{d,e,f\}\}$$

Set Theory - Power sets

If S is a set, then the *power set* of S is:

$$2^S = \{x : x \subseteq S\}$$

How is it denoted?

$$2^S \text{ or } P(S)$$

What is the cardinality?

$$2^{|S|}$$

$|S| = n$

$a_i \in S$

$\frac{1}{a_1} \quad \frac{1}{a_2} \quad \dots \quad \frac{0}{a_n}$ $= \{a_1, a_2\}$

Set Theory - Power sets

Which is these could be a power set?

$|\emptyset| = 0$

$\{\emptyset\}$ yes

$\emptyset$ no

$2^S = \{\emptyset, \{a\}\}$ yes $\quad S = \{a\}$

$\{\emptyset, \{a\}, \{\emptyset, a\}\}$ no

2^S $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ yes $\quad S = \{a, b\}$

$\{\emptyset, \{a\}, \{c\}, \{a, b\}\}$ NO

$S = \{a, b, c\}$

$2^S = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

Set Theory - Cartesian Product

Is A × B = B × A?

If $A \neq B$ then <u>no</u>

$A = \{a\}$ $\quad B = \{1, 2\}$

$A \times B = \{\langle a, 1\rangle, \langle a, 2\rangle\}$

$B \times A = \{\langle 1, a\rangle, \langle 2, a\rangle\}$

Set Theory - Cartesian Product

Is $A \times B \times C = A \times (B \times C)$? NO

$\langle x, y, z \rangle$

for $x \in A$
$y \in B$
$z \in C$

$\langle x, \langle y, z \rangle \rangle$

$A \times B \times C \neq$
$(A \times B) \times C$

Set Theory - Cartesian Product

How many elements will there be in:

$S = A_1 \times A_2 \times \ldots \times A_n$?

$$|S| = |A_1|\ |A_2| \ \ldots\ |A_n|$$

Set Theory - Definitions and notation

How would you show $A \subseteq B$?

- The key step is to pick a random element from A and show that it must be in B

Example

- For any sets A, B, C, and D, if $A \subseteq B$ and $C \subseteq D$ show

$$A \times C \subseteq B \times D$$

Set Theory - Definitions and notation

We need to show that if:

$$w \in A \times C \text{ then } w \in B \times D$$

What kind of thing is w?

$w = \langle x, y \rangle$

$x \in A$

$y \in C$

Set Theory - Definitions and notation

Let $w \in A \times C$, that means $w = \langle x,y \rangle$

with $x \in A$ and $y \in C$

Since $A \subseteq B$ then $x \in B$

Since $C \subseteq D$ then $y \in D$

So $(x,y) \in B \times D$

We have shown that for any $w \in A \times C$ it follows that

$w \in B \times D$, so $A \times C \subseteq B \times D$

Set Theory - Operators

What operators have we seen?

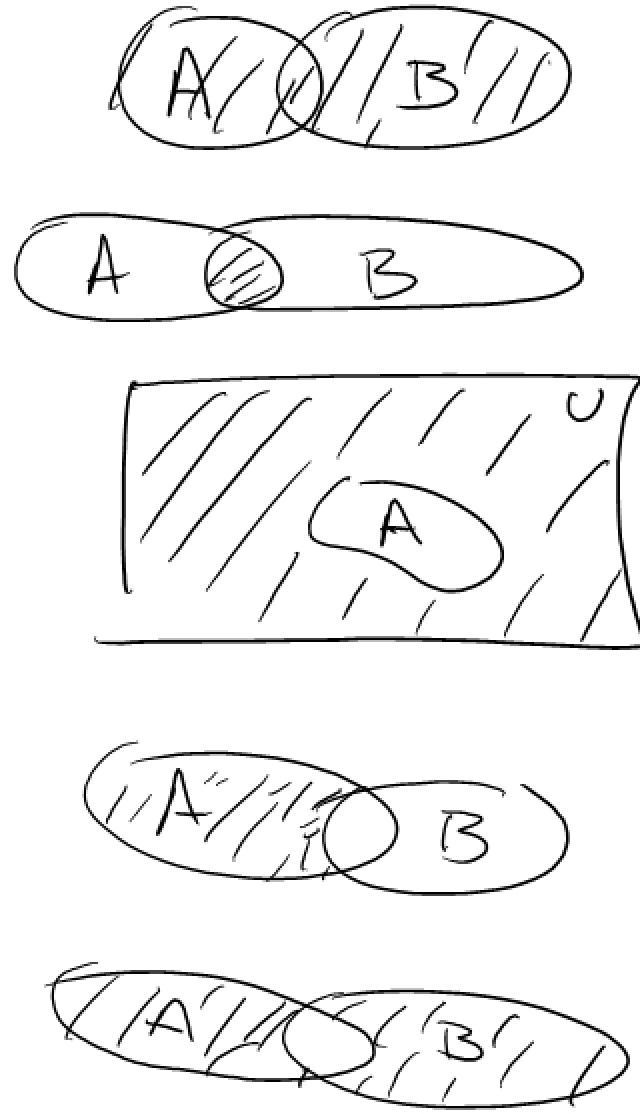
$A \cup B$

$A \cap B$

$\overline{A}$

$A - B$

$A \oplus B$



Set Theory - Operators

What operators have we seen?

Set Theory - Operators

What operators have we seen?

Set Theory - Operators

$$\boxed{A \oplus B} = \underbrace{\{ x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}}$$

$$= \boxed{(A - B) \cup (B - A)}$$

Proof:

$$\{ x : \underbrace{(x \in A \wedge x \notin B)} \vee (\underbrace{x \in B \wedge x \notin A})\}$$

$$= \{ x : \underbrace{(x \in A - B)} \vee \underbrace{(x \in B - A)}\}$$

$$= \{ x : \underbrace{x \in ((A - B) \cup (B - A))}\}$$

$$= \underline{(A - B) \cup (B - A)}$$

Set Theory - Famous Identities

- *Identity*

  $A \cap U = A$

  $A \cup \emptyset = A$

- *Domination*

  $A \cup U = U$

  $A \cap \emptyset = \cancel{A} \quad \emptyset$

- *Idempotent*

  $A \cup A = A$

  $A \cap A = A$

Set Theory - Famous Identities

- *Excluded Middle* $A \cup \bar{A} = U$
- *Uniqueness* $A \cap \bar{A} = \varnothing$
- *Double complement* $\bar{\bar{A}} = A$

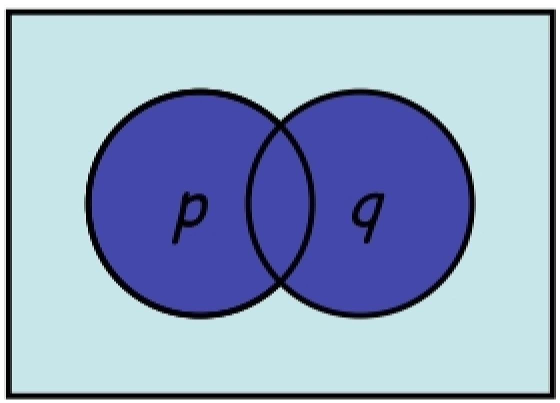
Set Theory - Famous Identities

- *Commutativity* $A \cup B = B \cup A$
  $A \cap B = B \cap A$
- *Associativity* $(A \cup B) \cup C = A \cup (B \cup C)$
  $(A \cap B) \cap C = A \cap (B \cap C)$
- *Distributivity* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Set Theory - Famous Identities

- *DeMorgan's I* $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

- *DeMorgan's II* $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$



Set Theory - 4 Ways to prove identities

- Show that $A \subseteq B$ and that $A \supseteq B$.
- Use a membership table.
- Use previously proven identities.
- Use logical equivalences to prove equivalent set definitions.

saw that with $A \oplus B = (B - A) \cup (A - B)$

Set Theory - 4 Ways to prove identities

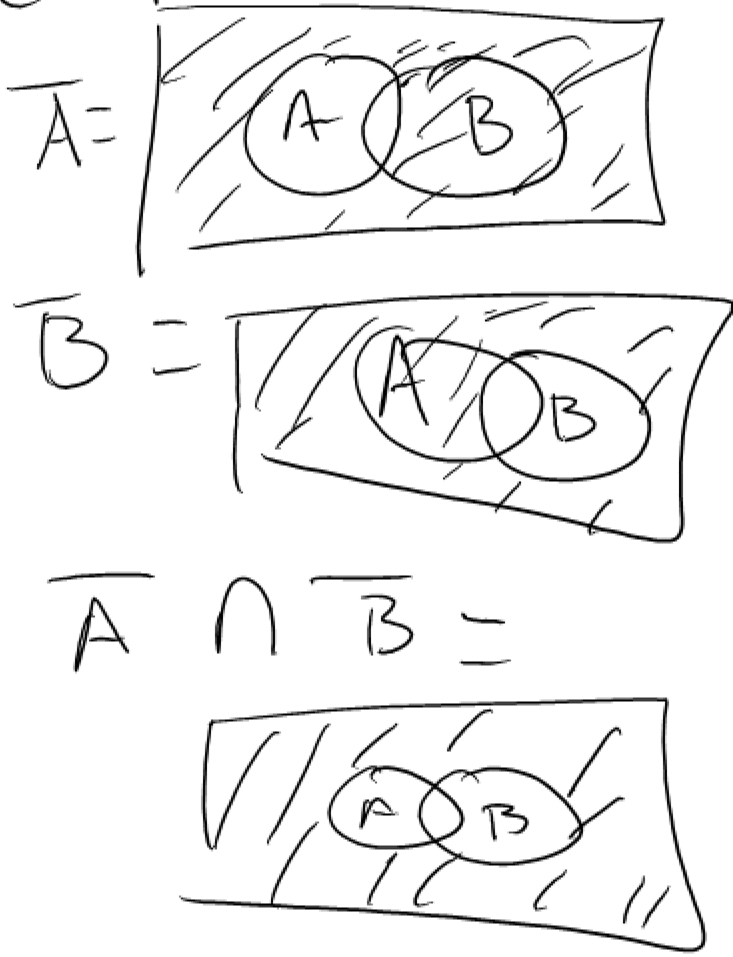
Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

by def of complement

1. $(\subseteq)$ $(x \in \overline{A \cup B}) \rightarrow (x \notin A \cup B) \rightarrow$
   $(x \notin A \text{ and } x \notin B) \rightarrow (x \in \overline{A} \cap \overline{B})$

2. $(\supseteq)$ $(x \in \overline{A} \cap \overline{B}) \rightarrow (x \notin A \text{ and } x \notin B)$
   $\rightarrow$ $(x \notin A \cup B) \rightarrow (x \in \overline{A \cup B})$

$A \subseteq B$ but $A \nsubseteq B$



Set Theory - 4 Ways to prove identities

Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ using a membership table.

0 : x is not in the specified set

1 : otherwise

A	B	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$	$A \cup B$	$\overline{A \cup B}$
1	1	0	0	0	1	0
1	0	0	1	0	1	0
0	1	1	0	0	1	0
0	0	1	1	1	0	1

Set Theory - 4 Ways to prove identities

Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ using identities.

$$\overline{(A \cup B)} = \overline{\overline{\overline{A}} \cup \overline{\overline{B}}}$$

$$= \overline{\overline{\overline{A} \cap \overline{B}}}$$

$$= \overline{A} \cap \overline{B}$$

Set Theory - 4 Ways to prove identities

Prove that $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ using logically equivalent set definitions.

$$\overline{(A \cup B)} = \{x : \neg(x \in A \vee x \in B)\}$$

$$= \{x : \neg(x \in A) \wedge \neg(x \in B)\}$$

$$= \{x : (x \in \overline{A}) \wedge (x \in \overline{B})\}$$

$$= \overline{A} \cap \overline{B}$$

Set Theory

$X \cap (Y - Z) = (X \cap Y) - (X \cap Z)$. True or False? Prove your response.

$(X \cap Y) - (X \cap Z) = (X \cap Y) \cap (X \cap Z)'$

$= (X \cap Y) \cap (X' \cup Z')$

$= (X \cap Y \cap X') \cup (X \cap Y \cap Z')$

$= \varnothing \cup (X \cap Y \cap Z')$

$= (X \cap Y \cap Z')$

Set Theory

Pv that if $(A - B) \cup (B - A) = (A \cup B)$ then

a) $A \cup B = \varnothing$

b) $A = B$

c) $A \cap B = \varnothing$

d) $A - B = B - A = \varnothing$

Set Theory

Pv that if $(A - B) \cup (B - A) = (A \cup B)$ then

a) $A \cup B = \emptyset$
b) $A = B$
c) $A \cap B = \emptyset$
d) $A-B = B-A = \emptyset$

Suppose to the contrary, that $A \cap B \neq \emptyset$, and that $x \in A \cap B$.

Then x cannot be in A-B and x cannot be in B-A.

DeMorgan's!!

Then x is not in $(A - B) \cup (B - A)$.

But x is in $A \cup B$ since $(A \cap B) \subseteq (A \cup B)$.

Thus, $A \cap B = \emptyset$.

Trying to pv p --> q

Assume p and not q, and find a contradiction.

Our contradiction was that sets weren't equal.