CS 173: Discrete Mathematical Structures, Spring 2009
Honors Homework 4:
Equivalence Classes and Permutation Groups

Due by 4pm on Wednesday 6 May. Please give to Margaret or push it under the door of her office (3214 Siebel).

This homework covers some more advanced counting techniques. Specifically we'll look at some problems where the number of solutions is reduced due to the symmetry of the problem. The mathematician George Polya developed the theory behind these counting techniques in 1938. We won't cover his entire theory, but we will go over permutation groups, Burnside's Theorem and some counting problems involving symmetry.

Before doing these problems, pick up a set of photocopied papers from outside room 2215 SC and read them – they explain the ideas behind permutation groups and Burnside's Theorem.

1. **Beads on a Necklace [15 points]** Count the number of distinct ways to arrange beads on a necklace, where there are 3 different colors of beads, and 3 total beads arranged on the necklace. The same color can appear on more than one bead (or it could appear on none). The symmetry of the necklace reduces the number of distinct colorings. With a necklace, we can obviously rotate it around. For simplicity, assume the necklace cannot be flipped over. You can see that symmetry reduces the number of distinct colorings by comparing a necklace colored "red, blue, red" with one colored "red, red, blue". Those two colorings are identical when rotated. We will count the number of distinct colorings of the necklace using Burnside's Theorem from the reading. This involves counting the number of colorings that are *invariant* under a given permutation. For example, one possible permutation of the necklace would be rotating the beads clockwise one position. Under this rotation, the coloring "red, blue, blue" would become "blue,red,blue". The coloring"red, red, red" is invariant under that permutation because the necklace looks the same after the permutation as it did before the permutation.

   (a) How many distinct colorings of the necklace are there when rotational equivalence is ignored?

   (b) Define a permutation group for the necklace $G = \{\pi_0, \pi_1, \pi_2\}$ where the permutations are clockwise rotations of the necklace and π_i is the permutation produced by i clockwise rotations. For example, if we denote the 3 beads as a, b, c, then $\pi_1 = \begin{pmatrix} abc \\ cab \end{pmatrix}$, which is a permutation based on one clockwise rotation of the necklace. Bead a moves to the position formerly occupied by bead b, bead b move to the spot where bead c was, and so on. For each π_i, what is the number of colorings of the necklace that are invariant under that permutation? *hint: the answer for π_0 is the same as the answer to part a of this question.*

(c) Using Burnside's Theorem, how many distinct colorings of the necklace exist?

(d) How many distinct colorings would exist if there were 6 beads?

2. **Coloring a Triangle [20 points]** Imagine we can color the vertices of an equilateral triangle any of three colors. We will call the three vertices of the triangle a, b, and c. Rotating and flipping the triangle is allowed. A rotation or flip is a permutation of the vertices. Note that a flip reflects the triangle across a line through one of the vertices that bisects the edge opposite that vertex.

   (a) How many 3-colored triangles are there when equivalence due to symmetry is ignored?

   (b) What is the permutation group for the triangle under these operations? Explicitly write out each permutation in the group, using the same notation that the photocopied pages employ. For example, write $\begin{pmatrix} abc \\ acb \end{pmatrix}$ to indicate a permutation mapping a to a, b to c and c to b.

   (c) By counting the triangle colorings that are invariant under each permutation and using Burnside's Theorem, how many distinct colorings of the triangle exist?

3. **Coloring a Chess Board [5 points]** Consider the 2×2 chess board described in the reading. Imagine the 4 squares of the board are labeled a, b, c, d starting with a in the top left corner and proceeding clockwise around the board. We are interested in the number of distinct ways to color the squares *white* or *black*. We can formalize this be considering functions that map the set of squares $\{a, b, c, d\}$ to the set of colors $\{b, w\}$. In counting distinct colorings, we will consider permutations based on rotating the board by 90 degrees. There are 4 such permutations in the permutation group

$$G = \left\{ \pi_0 = \begin{pmatrix} abcd \\ abcd \end{pmatrix}, \pi_1 = \begin{pmatrix} abcd \\ dabc \end{pmatrix}, \pi_2 = \begin{pmatrix} abcd \\ cdab \end{pmatrix}, \pi_3 = \begin{pmatrix} abcd \\ bcda \end{pmatrix} \right\}.$$

   We consider two functions f and g to be equivalent if there is a permutation π_i such that for all squares s, $f(\pi_i(s)) = g(s)$. For example, consider a function f_3 that maps the board (a, b, c, d) to the colors (w, b, w, w) and another function f_2 that maps (a, b, c, d) to (b, w, w, w). Under the permutation π_3 you can see that f_3 is equivalent to f_2.

   (a) Write out a table of the 16 functions mapping squares to colors.

   (b) The permutation group G divides the 16 functions into 6 equivalence classes. List these classes (e.g. f_3 and f_2 from above would both be in the same class) .

4. **Equivalence Classes of Functions [10 points]** Suppose D and R are two sets and let G be a permutation group of the set D. We define a binary relation on the set of all functions from D to R. A function f_1 is related to a function f_2 if and only if there is a permutation π in G such that $f_1(\pi(d)) = f_2(d)$ for all d in D. In other words, just as in Problem 3, we are defining a notion of two functions being equivalent under a permutation π if f_1 maps $\pi(d)$ to the same thing as f_2 maps d for all d in D. Prove this binary relation is an equivalence relation.