CS 173, Spring 2009
Homework 8

Due in class on Friday, April 17th, 2009
(Total point value: 50 points.)

1. **Counting I [10 points]** For the following four questions, you do not need to multiply out factorials to reach a final answer. For example, $P(10,4) = \frac{10!}{6!}$ would be acceptable as an answer; you don't have to complete the multiplications and division.

   (a) Suppose a set S has 10 elements, how many subsets of S have an odd number of elements?

   (b) How many bit-strings of length 100 have exactly 10 zeroes?

   (c) How many distinct strings can be formed by the letters in the word BOOTHBAY?

   (d) Suppose that after taking a job at Initech you have 7 managers, each of whom sends you one memo per day. Initech memos come in three types: secret, company internal, and already reported by CNET ("public" for short). How many different combinations of memo types could you receive in one day? (E.g. one combination would be 1 secret, 5 internal, and 1 public, which is different from the combination 2 secret, 1 internal, and 4 public.)

2. **Counting II [10 points]** For the following two questions, you do not need to multiply out factorials to reach a final answer.

   (a) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 17$ when x_i is a non-negative integer for $1 \leq i \leq 4$.

   (b) The field of bioinformatics makes use of discrete mathematics in many applications. We will consider the problem of counting the number of ways a certain molecule can be constructed. RNA, or *ribonucleic acid*, is a long molecule that is used by some cells to transfer information. RNA is essentially a chain of bases in which each base is either adenine (**A**), urasil (**U**), guanine (**G**), or cytosine(**C**). In an RNA chain of 20 bases, suppose there are 4 **A**s, 5 **U**s, 6 **G**s, and 5 **C**s. If the chain must begin with either **AC** or **UG**, how many such chains are there?

3. **Counting Proofs [10 points]**

   (a) Prove that following formula holds, for any k and n with $n > k \geq 0$.

$$\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} = \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}$$

   (b) Prove that the following holds for any integer $n \geq 2$:

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

4. **Structural induction [10 points]**

   Define a set $M \subseteq \mathbb{Z}^2$ as follows

   (1) $(3, 2) \in M$

   (2) If $(x, y) \in M$, then $(3x - 2y, x) \in M$

   Use structural induction to prove that elements of M always have the form $(2^{k+1} + 1, 2^k + 1)$, where k is a natural number. (The point of this problem is to learn how to use structural induction, so you may not rephrase this into a normal proof by induction on k.)

5. **Tree induction [10 points]**

   The Fibonacci trees T_k are a special sort of binary trees that are defined as follows.

   Base: T_1 and T_2 are binary trees with only a single vertex.

   Induction: For any $n \geq 3$, T_n consists of a root node with T_{n-1} as its left subtree and T_{n-2} as its right subtree.

   Use structural induction to prove that the height of T_n is $n - 2$, for any $n \geq 2$. (Again, use structural induction rather than looking for an explicit induction variable n.)