# CS 173, Spring 2009 <br> Homework 8 Solutions 

## (Total point value: 50 points.)

1. Counting I [10 points] For the following four questions, you do not need to multiply out factorials to reach a final answer. For example, $P(10,4)=\frac{10!}{6!}$ would be acceptable as an answer; you don't have to complete the multiplications and division.
(a) Suppose a set $S$ has 10 elements, how many subsets of $S$ have an odd number of elements?
Solution. Subsets of $S$ having an odd number of elements would have $1,3,5,7$, or 9 elements. There are $C(10,1)=10$ subsets of $S$ with one element, $C(10,3)=\frac{10!}{7!3!}$ subsets of $S$ with 3 elements, and so on. So the answer is the sum of all these cases. Therefore, the number of subsets of $S$ having an odd number of elements is: $\sum_{i=0}^{4} C(10,2 i+1)$
(b) How many bit-strings of length 100 have exactly 10 zeroes?

Solution. We can count these strings by choosing a subset of 10 positions that will contain the zeros. So the number of bit-strings of length 100 that have exactly 10 zeros is : $C(100,10)=\frac{100!}{90!10!}$.
(c) How many distinct strings can be formed by the letters in the word BOOTHBAY?

Solution. The number of strings that can be formed by the letters in the word BOOTHBAY, using all the letters is: $\frac{8!}{2!2!}$. (This uses the formula for a permutations with indistinguishable objects.)
(d) Suppose that after taking a job at Initech you have 7 managers, each of whom sends you one memo per day. Initech memos come in three types: secret, company internal, and already reported by CNET ("public" for short). How many different combinations of memo types could you receive in one day? (E.g. one combination would be 1 secret, 5 internal, and 1 public, which is different from the combination 2 secret, 1 internal, and 4 public.)
Solution. Since it is not important who sends which type of document, we can use the formula for combinations with repetition. So the number of different combinations is equal to: $C(3+7-1,7)=C(9,7)=\frac{9!}{7!2!}$.
2. Counting II [10 points] For the following two questions, you do not need to multiply out factorials to reach a final answer.
(a) How many solutions are there to the equation $x_{1}+x_{2}+x_{3}+x_{4}=17$ when $x_{i}$ is a non-negative integer for $1 \leq i \leq 4$.
Solution. This equation has $C(17+4-1,17)=\frac{20 \text { ! }}{17!3!}$ solutions. To see this, imagine the 17 as 17 identical objects. We need to divide them into four bins, i.e. place 3 dividers into the list of objects. See the pictures used to analyze combinations with repetition.
(b) The field of bioinformatics makes use of discrete mathematics in many applications. We will consider the problem of counting the number of ways a certain molecule can be constructed. RNA, or ribonucleic acid, is a long molecule that is used by some cells to transfer information. RNA is essentially a chain of bases in which each base is either
adenine (A), urasil (U), guanine (G), or cytosine(C). In an RNA chain of 20 bases, suppose there are $4 \mathrm{As}, 5 \mathrm{Us}, 6 \mathrm{Gs}$, and 5 Cs . If the chain must begin with either AC or UG, how many such chains are there?
Solution. Since the chains must begin with either AC or UG, we have two mutually exclusive cases. In either case, we need to choose 18 bases (because the first two are fixed), each of which can be an adenine (A), urasil (U), guanine (G), or cytosine(C). If the chain starts with AC, we have 3 As, 5 Us, 6 Gs, and 4 Cs left. Therefore, using the formula for combinations with repetition, there are $\frac{18!}{3!4!5!6!}$ possibilities. If it starts with UG, we have 4 As, 4 Us, 5 Gs , and 5 Cs left, so we have $\frac{18!}{5!5!4!4!}$ chains. Adding the two cases together, we find that there are $\frac{18!}{3!45!5!}+\frac{18!}{5!5!4!4!}$ number of chains.

## 3. Counting Proofs [10 points]

(a) Prove that following formula holds, for any $k$ and $n$ with $n>k \geq 0$.

$$
\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k}=\binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1}
$$

Solution. $\binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k}$ is equal to: $\frac{(n-1)!}{(k-1)!(n-k)!} \cdot \frac{n!}{(k+1)!(n-k-1)!}$. $\frac{(n+1)!}{k!(n+1-k)!}$.
By shuffling terms around on the bottoms of the fractions, we can rewrite this equation as: $\frac{(n-1)!}{k!(n-1-k)!} \cdot \frac{n!}{(k-1)!(n-k+1)!} \cdot \frac{(n+1)!}{(k+1)!(n-k)!}$,
which is equal to:

$$
\binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1} .
$$

(b) Prove that the following holds for any integer $n \geq 2$ :

$$
\binom{2 n}{2}=2\binom{n}{2}+n^{2}
$$

Solution.

$$
\begin{aligned}
\binom{2 n}{2}= & \frac{2 n!}{(2 n-2)!2!}=\frac{(2 n)(2 n-1)}{2}=2 n^{2}-n \\
= & n^{2}-n+n^{2}=n(n-1)+n^{2} \\
& =\frac{n(n-1)(n-2)!}{(n-2)!}+n^{2} \\
& =2 \frac{n(n-1)(n-2)!}{(n-2)!2!}+n^{2}
\end{aligned}
$$

$$
\begin{aligned}
= & 2 \frac{n!}{(n-2)!2!}+n^{2} \\
& =2\binom{n}{2}+n^{2}
\end{aligned}
$$

We can also prove this claim by induction:

## Proof by induction

Base case: For $n=2$

$$
\binom{2 n}{2}=\binom{4}{2}=6
$$

and

$$
2\binom{n}{2}+n^{2}=2\binom{2}{2}+2^{2}=6
$$

Induction hypothesis: Assume

$$
\binom{2 n}{2}=2\binom{n}{2}+n^{2}
$$

is true for $n$.
Induction step: Assuming the claim is true for $n$, we need to prove it is true for $n+1$. In other words, we need to show:

$$
\binom{2 n+2}{2}=2\binom{n+1}{2}+(n+1)^{2}
$$

First observe:

$$
\begin{gathered}
\binom{2 n+2}{2}=\binom{2 n+1}{1}+\binom{2 n+1}{2} \\
=(2 n+1)+\binom{2 n}{1}+\binom{2 n}{2} \\
=(2 n+1)+2 n+\binom{2 n}{2}
\end{gathered}
$$

Using the induction hypothesis we have:

$$
=4 n+1+2\binom{n}{2}+n^{2} .
$$

$$
\begin{gathered}
=n^{2}+(2 n+1)+2\left(n+\binom{n}{2}\right) \\
=n^{2}+(2 n+1)+2\left(\binom{n}{1}+\binom{n}{2}\right) \\
=n^{2}+(2 n+1)+2\left(\binom{n+1}{2}\right) \\
=2\left(\binom{n+1}{2}\right)+(n+1)^{2}
\end{gathered}
$$

and this proves our claim for $n+1$.

## 4. Structural induction [10 points]

Define a set $M \subseteq \mathbb{Z}^{2}$ as follows
(1) $(3,2) \in M$
(2) If $(x, y) \in M$, then $(3 x-2 y, x) \in M$

Use structural induction to prove that elements of $M$ always have the form $\left(2^{k+1}+1,2^{k}+1\right)$, where $k$ is a natural number. (The point of this problem is to learn how to use structural induction, so you may not rephrase this into a normal proof by induction on $k$.)

## Solution.

Base: $3=2^{0+1}+1$ and $2=2^{0}+1$. So the relationship holds for $(3,2)$.
Induction: Assume that for some $(x, y) \in M, x=2^{k+1}+1$, and $y=2^{k}+1$.
We must show that the property holds for $(3 x-2 y, x)$, in other words, that $3 x-2 y=2^{m+1}+1$, and $x=2^{m}+1$, for some integer $m$.
Based on the induction hypothesis we have:
$3 x-2 y=3\left(2^{k+1}+1\right)-2\left(2^{k}+1\right)=3 \cdot 2^{k+1}+3-2^{k+1}-2$
$=2 \cdot 2^{k+1}+1=2^{k+2}+1$
Now if we choose $m$ to be $k+1$, we have $3 x-2 y=2^{k+2}+1=2^{m+1}+1$ and $x=2^{k+1}+1=$ $2^{m}+1$. So we have proved the claim.

## 5. Tree induction [10 points]

The Fibonacci trees $T_{k}$ are a special sort of binary trees that are defined as follows.
Base: $T_{1}$ and $T_{2}$ are binary trees with only a single vertex.
Induction: For any $n \geq 3, T_{n}$ consists of a root node with $T_{n-1}$ as its left subtree and $T_{n-2}$ as its right subtree.

Use structural induction to prove that the height of $T_{n}$ is $n-2$, for any $n \geq 2$. (Again, use structural induction rather than looking for an explicit induction variable $n$.)
Solution. Proof by structural induction.
Base: $T_{2}$ is a Fibonacci tree with a single vertex and no edges and therefore has height 0 . $n-2$ is also equal to 0 .
Induction: Suppose that the claim is true for $T_{k}, k<n$. We need to show that the claim is also true for $T_{n}$.
$T_{n}$ consists of a root node with daughters $T_{n-1}$ and $T_{n-2}$. By the inductive hypothesis, the claim holds for $T_{n-1}$ and $T_{n-2}$, so we know that the height of $T_{n-2}$ is $n-3$ and the height of $T_{n-2}$ is $n-4$.
Now, we know that the height of any binary tree is one more than the maximum height of any of its daughter trees. So we have: height $\left(T_{n}\right)=\max \left(\operatorname{height}\left(T_{n-1}\right)\right.$, $\left.\operatorname{height}\left(T_{n-2}\right)\right)+1$. Substituting in the heights of $T_{n-1}$ and $T_{n-2}$, we find that $\operatorname{height}\left(T_{n}\right)=(n-3)+1=n-2$, and this proves the claim.

