CS 173, Spring 2009
Homework 6 Solution

Total point value: 50 points.

1. **Recursive definition of a set [10 points]**

   (a) Define the set $S \subseteq \mathbb{Z}^2$ as follows:

   rule 1: $(0,0) \in S$
   rule 2: $(10,0) \in S$
   rule 3: If $(x,y) \in S$, then $(y,x) \in S$
   rule 4: If $(x,y) \in S$, then $(-x,y) \in S$

   What points does S contain?

   Solution:

   S={ (0,0), (10,0), (0,10),(-10,0),(0,-10)} This gives the four corner points of a square.

   (b) Suppose that (x,y) and (p,q) are 2D points, whose coordinates might not necessarily be integers. Describe in words how the 2D point $(\frac{x+p}{2}, \frac{y+q}{2})$ is geometrically related to (x,y) and (p,q).

   Solution:

   The 2D point $(\frac{x+p}{2}, \frac{y+q}{2})$ is nearest integer point with coordinates smaller than the mid-point of the line segment joining the two 2D points (x,y) and (p,q).

   (c) Suppose we define a set T using rules 1 and 2 above, plus the following rule 5 (but not rules 3 and 4). What points does T contain?

   rule 5: If $(x,y) \in S$ and $(p,q) \in S$, then $(\lfloor \frac{x+p}{2} \rfloor, \lfloor \frac{y+q}{2} \rfloor) \in S$

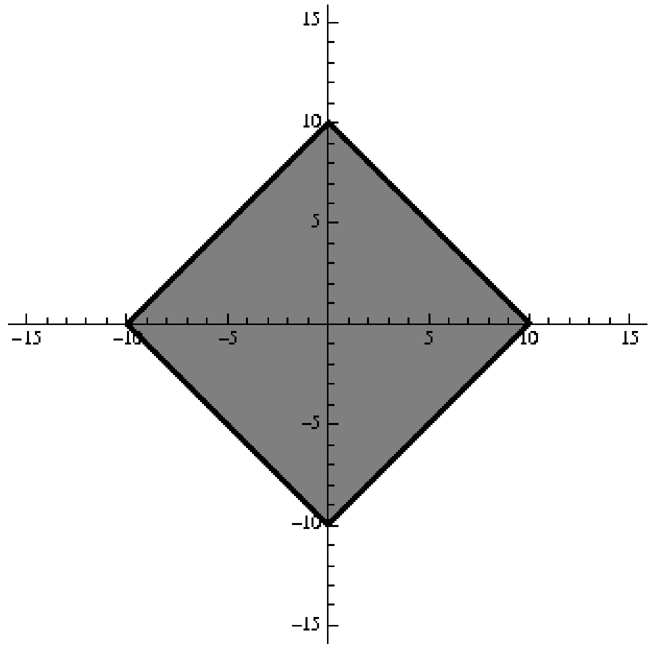
   Solution:

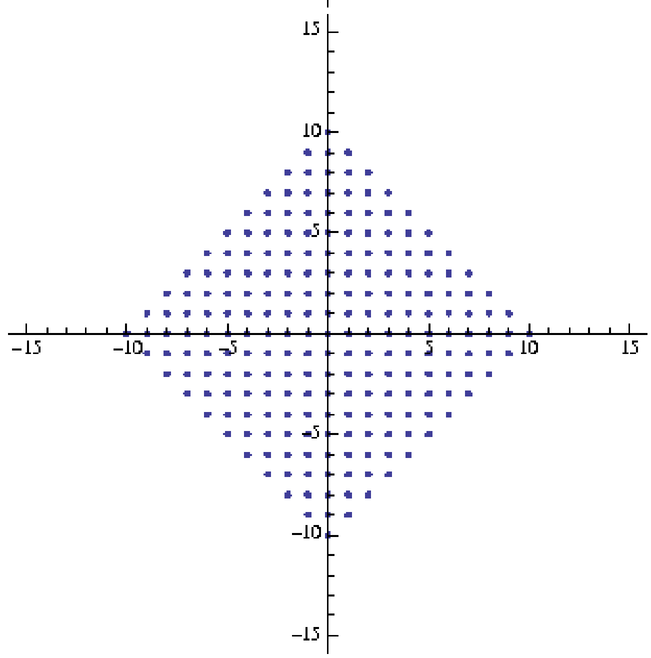
   T={(0,0),(10,0), (5,0), (2,0), (7,0), (1,0),(3,0), (6,0), (8,0),(4,0), (9,0)}

   Thus S has all the points (with integer coordinates) on the x-axis in between the points $(0,0)$ and $(10,0)$. We can also say that this set contains the line segment between the two points $(0,0)$ and $(10,0)$. Basically for any two points in S, rule 5 adds the line segment in between them. (Integer points only, in both cases.)

   (d) Suppose that we define a set R using all five rules. Give a picture and a succinct, closed-form description of the set R, showing work and/or briefly justifying your answer.

   Here are two ways to draw the picture, depending on whether you prefer to draw all the dots or just say in words that it's only the integer pairs within the filled region.





We see that from rules 1-4, we get the corner points of a square. Then applying rule 5 we add the points in between any pair of elements in the set. Hence, we see that we basically get all the points surrounded by the perimeter of the square. Hence, R is the area of a square, where by area we mean all the points (with integer coordinates) within and on the boundary of that area.

Or... at least this is what we intended the answer to be. And this answer is worth full credit. As it turns out, the floor operation has unintended consequences. If you apply rule 5 to $(-5, -5)$ and $(0, -10)$, you can get $(-3, -8)$, which is just outside the boundary of the square. This only happens in the negative quadrant. But if you keep combining negative points, you can eventually extend that part of the figure out to $(-10, -10)$. And, then, the full set expands to all the integer pairs in the entire square with corners $(10, 10)$, $(-10, 10)$, $(10, -10)$, and $(-10, -10)$. And then the process really does stop.

2. **Functions with sets [10 points]**

Define two functions g and f as follows, for all positive integer inputs.

$$g(1) = \{1\}$$
$$g(n) = g(n-1) \cup \{n\}$$

$$f(1) = \{1\}$$
$$f(n) = f(n-1) \times g(n)$$

(a) The inputs to f and g are positive integers, so the domain for each of them is $\mathbb{Z}^+$. What sort of objects do g and f produce as output and, therefore, what is the co-domain for each function?

Solution:

g produces sets of positive integers. So the co-domain of g can be defined as $\{1, \ldots, x | x \in \mathbb{Z}^+\}$ which contains some, but not all subsets of $\mathbb{Z}^+$. You could also choose the co-domain to be an over-estimate of this set, e.g. $\mathbb{P}(\mathbb{Z}^+)$ or $\mathbb{P}(\mathbb{Z})$.

f produces cartesian products of sets which can be defined as $\{((\phi \times g(1)) \times g(2)) \times \ldots) \ldots \times g(x) | x \in \mathbb{Z}^+\}$. So we can write the co-domain of f as $\mathbb{Z}^+ \cup (\mathbb{Z}^+ \times \mathbb{Z}^+) \cup ((\mathbb{Z}^+ \times \mathbb{Z}^+) \times \mathbb{Z}^+) \cup \ldots$. We haven't yet seen a succinct notation for sets like this. Again, it's ok to over-estimate this set e.g. $\mathbb{Z} \cup (\mathbb{Z} \times \mathbb{Z}) \cup ((\mathbb{Z} \times \mathbb{Z}) \times \mathbb{Z}) \cup \ldots$.

If we want to be a bit less formal about the pair structure, we could say that f produces integers, pairs of integers, triples of integers, and in general n-tuples of integers. So the co-domain is the union of $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, and so forth. This isn't quite right technically, but it's a common simplification.

(b) Compute the value $g(5)$.

$$g(1) = \{1\},\ g(2) = \{1, 2\},\ g(3) = \{1, 2, 3\},\ g(4) = \{1, 2, 3, 4\},\ g(5) = \{1, 2, 3, 4, 5\}$$

(c) Compute the value $f(3)$

$$\begin{aligned}
f(1) &= \{1\} \\
f(2) &= f(1) \times g(2) \\
&= \{(1,1), (1,2)\} \\
f(3) &= f(2) \times g(3) \\
&= \{(1,1), (1,2)\} \times \{1, 2, 3\} \\
&= \{((1,1),1), ((1,2),1), ((1,1),2), ((1,2),2), ((1,1),3), ((1,2),3)\}
\end{aligned}$$

(d) Give a closed-form formula for $|f(n)|$, as a function of n.

Solution:

$|f(n)| = n!$

3. **Big-O [10 points]**

For each of the following pairs of functions state whether $f(n) = O(g(n))$ or $f(n) = \Omega(g(n))$ or $f(n) = \Theta(g(n))$

(a) $f(n) = \lceil n \rceil^2$ and $g(n) = \lfloor n \rfloor^2$.

Solution: $f(n) = \Theta(g(n))$. The two functions are equal for integer inputs. For non-integer inputs, floor and ceiling can differ by at most 1. So the difference between the squares will be no more than $(n^2 + 2n + 1) - n^2 = 2n + 1$. As n gets large, this grows more slowly than n^2.

(b) $f(n) = (log_{10}(n))^2$ and $g(n) = n$. The log function grows more slowly than n, and the square of the log function grows even more slowly.

Solution: f(n)=O(g(n)).

(c) $f(n) = n^{2^n}$ and $g(n) = n^{n^2}$, because 2^n grows much faster than n^2.

Solution: $f(n) = \Omega(g(n))$.

(d) $f(n) = n!$ and $g(n) = n^n$.

Solution: $f(n) = O(g(n))$. If you expand both functions into products, their first terms are the same, but then the later terms are much smaller for $n!$. This difference gets bigger as n gets larger.

(e) $f(n) = 2^n + n$ and $g(n) = 3^n$

Solution: $f(n) = O(g(n))$. You can ignore the n term in f. As n gets large, the difference between 2^n and 3^n gets larger, because you keep multiplying in more factors of $\frac{3}{2}$.

Determine whether each statement below is true or false.

(f) If $f(n) = \Theta(g(n))$ and $h(n) = \Theta(g(n))$ then $f(n)h(n) = \Theta(g(n))$.

Solution: False. $f(n)h(n) = 0(g(n)g(n)) \rightarrow \forall x > k, f(x)h(x) \leq cg(x)g(x)$. $cg(x)$ cannot be replaced by a constant.

(g) If $f(n) = \Omega(g(n))$ and $h(n) = \Omega(g(n))$ then $f(n) + h(n) = \Omega(g(n))$.

Solution: True. We have, $f(x) \geq c_1 g(x)$ for all $x > k_1$ and $h(x) \geq c_2 g(x)$ for all $x > k_2$. Hence we have that $f(x)+h(x) \geq c_1 g(x)+c_2 g(x) = (c_1+c_2)g(x)$ for all $x > max(k_1, k_2)$.

(h) If $f(n) = O(g(n))$ then $g(n) = \Omega(f(n))$.

Solution: True. We saw this in lecture. (It's one way to define Ω.)

(i) If $f(n) = \Theta(g(n))$ then $g(n) = \Theta(f(n))$

Solution: True. Θ is a type of equality. If it's true in one direction, it's true in the other.

(j) If $f(n) = log_a(n)$ for $a > 2$ then $f(n) \neq \Theta(log_2(n))$.

Solution: False. Changing the base of $\log_a n$ multiplies all the outputs by a number that doesn't depend on n (it only depends on the old and new bases).

When the right answer is given as $f(n) = \Theta(g(n))$ above, you could also (optionally) say that $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

4. **Big-O proofs [10 points]**

Prove the following statements

(a) Prove that $f(n) = n^n$ is **not** $O(2^n)$ *(hint: use proof by contradiction)*.

Solution Let us assume that $f(n) = n^n$ is $O(2^n)$. This implies that there are integers c and k such that, for all $n > k$, we have that,

$$
\begin{aligned}
& n^n \le c2^n \\
\Rightarrow\quad & n \log_2 n \le \log_2 c + n \log_2 2 \\
\Rightarrow\quad & n \log_2 n \le \log_2 c + n \\
\Rightarrow\quad & n \log_2 n - \log_2 c \le n \\
\Rightarrow\quad & n(\log_2 n - 1) \le \log_2 c
\end{aligned}
$$

Suppose we pick n so that it is larger than 4 and also larger than $\log_2 c$. Then then $\log_2 n - 1 > 1$. So $n(\log_2 n - 1) > n > \log_2 c$. We've now derived two contradictory inequalities, so this contradicts our assumption, and so it is not the case that $f(n) = n^n$ is $O(2^n)$.

Here's another way to do the algebra. Let d be the larger of c and 1. We have that $n^n \le c2^n$, so $n^n \le d2^n$, so So then $n^n \le d^n 2^n = (2d)^n$. If n is at least 1, this is true exactly when $n \le 2d$. But this clearly fails if we pick a large enough value for n.

(b) Prove that $f(n) = n^2 + 8n + 2$ is $\Theta(n^2)$.

Solution: We need to prove that $f(n) = n^2 + 8n + 2$ is $O(n^2)$ and $f(n) = n^2 + 8n + 2$ is $\Omega(n^2)$.

For the first part, we notice that $2 < n^2$ and $8n < n^2$ when $n > 8$. It follows that $0 \le n^2 + 8n + 2 \le n^2 + n^2 + n^2 = 3n^2$ whenever, $n > 8$. So using $c = 3$ and $k = 8$ as witnesses, we get that $f(n)$ is $O(n^2)$.

For the second part, we notice that $n^2 + 8n + 2 \ge n^2$ for all $n > 1$. Hence we can use $c = 1$ and $k = 1$ as witnesses and get that, $f(n) = \Omega(n^2)$.

Since we have shown that $f(n) = O(n^2)$ and $f(n) = \Omega(n^2)$, we have that $f(n) = \Theta(n^2)$.

5. **Induction [10 points]**

Define the function f as follows:

- $f(1) = 1$
- $f(2) = 5$
- $f(n+1) = 5f(n) - 6f(n-1)$

(a) Compute $f(3)$ and $f(4)$.

Solution:

$$
\begin{aligned}
f(3) &= 5f(2) - 6f(1) \\
&= 5 * 5 - 6 * 1 \\
&= 19
\end{aligned}
$$

$$
\begin{aligned}
f(4) &= 5f(3) - 6f(2) \\
&= 5 * 19 - 6 * 5 \\
&= 65
\end{aligned}
$$

(b) Use strong induction to prove that $f(n) = 3^n - 2^n$ for every positive integer n.

Solution:

Let $P(n)$ be the proposition that $f(n) = 3^n - 2^n$ for every positive integer n. We will prove it by strong induction over n.

Base Step: The base cases are when $n = 1$ and $n = 2$. When $n = 1$ we have $3^1 - 2^1 = 3 - 2 = 1$ and when $n = 2$, we have, $3^2 - 2^2 = 9 - 4 = 5$ which matches the given values of $f(1)$ and $f(2)$.

Inductive Step: We assume that $P(j)$ holds for $1 \leq j \leq k$. Using this inductive hypothesis, we will now show that $P(k+1)$ holds.

We have,

$$\begin{aligned}
f(k+1) &= 5f(k) - 6f(k-1) \\
&= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1}) \\
&= 5 * 3^k - 5 * 2^k - 2 * 3 * 3^{k-1} + 2 * 3 * 2^{k-1} \\
&= 5 * 3^k - 5 * 2^k - 2 * 3^k + 3 * 2^k \\
&= 3 * 3^k - 2 * 2^k \\
&= 3^{k+1} - 2^{k+1}
\end{aligned}$$

Thus we have that $P(k+1)$ holds. This completes our proof.