CS 173, Spring 2009
Homework 6

Due in class on Friday, March 20th, 2009
(Total point value: 50 points.)
If you are leaving town early, remember that you can push your homework
under the door of Margaret's office (3214 Siebel).

1. **Recursive definition of a set [10 points]**

   (a) Define the set $S \subseteq \mathbb{Z}^2$ as follows:

   rule 1: $(0,0) \in S$
   rule 2: $(10,0) \in S$
   rule 3: If $(x,y) \in S$, then $(y,x) \in S$
   rule 4: If $(x,y) \in S$, then $(-x,y) \in S$

   What points does S contain?

   (b) Suppose that (x,y) and (p,q) are 2D points, whose coordinates might not necessarily be integers. Describe in words how the 2D point $(\frac{x+p}{2}, \frac{y+q}{2})$ is geometrically related to (x,y) and (p,q).

   (c) Suppose we define a set T using rules 1 and 2 above, plus the following rule 5 (but not rules 3 and 4). What points does T contain?

   rule 5: If $(x,y) \in S$ and $(p,q) \in S$, then $(\lfloor \frac{x+p}{2} \rfloor, \lfloor \frac{y+q}{2} \rfloor) \in S$

   (d) Suppose that we define a set R using all five rules. Give a picture and a succinct, closed-form description of the set R, showing work and/or briefly justifying your answer.

2. **Functions with sets [10 points]**

   Define two functions g and f as follows, for all positive integer inputs.

   $g(1) = \{1\}$
   $g(n) = g(n-1) \cup \{n\}$

   $f(1) = \{1\}$
   $f(n) = f(n-1) \times g(n)$

   (a) The inputs to f and g are integers, so the domain for each of them is $\mathbb{Z}$. What sort of objects do g and f produce as output and, therefore, what is the co-domain for each function?

   (b) Compute the value $g(5)$.

   (c) Compute the value $f(3)$

   (d) Give a closed-form formula for $|f(n)|$, as a function of n.

3. **Big-O [10 points]**

   For each of the following pairs of functions state whether $f(n) = O(g(n))$ or $f(n) = \Omega(g(n))$ or $f(n) = \Theta(g(n))$

(a) $f(n) = \lceil n \rceil^2$ and $g(n) = \lfloor n \rfloor^2$.

(b) $f(n) = (log_{10}(n))^2$ and $g(n) = n$.

(c) $f(n) = n^{2^n}$ and $g(n) = n^{n^2}$

(d) $f(n) = n!$ and $g(n) = n^n$

(e) $f(n) = 2^n + n$ and $g(n) = 3^n$

Determine whether each statement below is true or false.

(f) If $f(n) = \Theta(g(n))$ and $h(n) = \Theta(g(n))$ then $f(n)h(n) = \Theta(g(n))$.

(g) If $f(n) = \Omega(g(n))$ and $h(n) = \Omega(g(n))$ then $f(n) + h(n) = \Omega(g(n))$.

(h) If $f(n) = O(g(n))$ then $g(n) = \Omega(f(n))$.

(i) If $f(n) = \Theta(g(n))$ then $g(n) = \Theta(f(n))$

(j) If $f(n) = log_a(n)$ for $a > 2$ then $f(n) \neq \Theta(log_2(n))$.

4. **Big-O proofs [10 points]**

Prove the following statements

(a) Prove that $f(n) = n^n$ is **not** $O(2^n)$ *(hint: use proof by contradiction)*.

(b) Prove that $f(n) = n^2 + 8n + 2$ is $\Theta(n^2)$. *(hint: remember to prove both halves of this claim)*

5. **Induction [10 points]**

Define the function f as follows:

- $f(1) = 1$
- $f(2) = 5$
- $f(n+1) = 5f(n) - 6f(n-1)$

(a) Compute $f(3)$ and $f(4)$.

(b) Use strong induction to prove that $f(n) = 3^n - 2^n$ for every positive integer n.