CS 173, Spring 2009
Homework 5 Solutions

(Total point value: 50 points.)

1. **Recursive definition [10 points]**

   (a) Consider the function h defined by the following recursive definition. Compute $h(x)$ for x from 0 to 10.

   $h(0) = 0$
   $h(1) = 1$
   $h(2) = 1$
   $h(n) = h(n-1) + h(n-2) - h(n-3)$ if $n \geq 3$

   [Solution]

   $h(0) = 0$
   $h(1) = 1$
   $h(2) = 1$
   $h(3) = 1 + 1 - 0 = 2$
   $h(4) = 2 + 1 - 1 = 2$
   $h(5) = 2 + 2 - 1 = 3$
   $h(6) = 3 + 2 - 2 = 3$
   $h(7) = 3 + 3 - 2 = 4$
   $h(8) = 4 + 3 - 3 = 4$
   $h(9) = 4 + 4 - 3 = 5$
   $h(10) = 5 + 4 - 4 = 5$

   (b) Consider the function g defined by the following recursive definition. Compute $g(x)$ for x from 1 to 10.

   $g(1) = 1$
   $g(2) = 2$
   $g(n) = g(n-1) + g(n-2)$ if $n \geq 3$

   [Solution]

   $g(1) = 1$
   $g(2) = 2$
   $g(3) = 2 + 1 = 3$
   $g(4) = 3 + 2 = 5$
   $g(5) = 5 + 3 = 8$
   $g(6) = 8 + 5 = 13$
   $g(7) = 13 + 8 = 21$
   $g(8) = 21 + 13 = 34$
   $g(9) = 34 + 21 = 55$
   $g(10) = 55 + 34 = 89$

2. **Induction [10 points]**

   Use induction to prove that the following formula holds, for any positive integer n.

   $$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$$

   [Solution]

   We will show this by induction on n.

   Base: Consider when $n = 1$: $\sum_{k=1}^{1} k2^k = 1 \cdot 2^1 = 2$ and $(1-1)2^{1+1} + 2 = 2$. Since these are equal, the formula holds for $n = 1$.

   Induction: Suppose that the claim holds for $n = i$. That is, there is an $i \in \mathbb{Z}^+$ such that $\sum_{k=1}^{i} k2^k = (i-1)2^{i+1} + 2$. We need to show: $\sum_{k=1}^{i+1} k2^k = ((i+1)-1)2^{(i+1)+1} + 2 = i2^{i+2} + 2$

   By pulling a term out of the summation, we can write $\sum_{k=1}^{i+1} k2^k$ as $\left(\sum_{k=1}^{i} k2^k \right) + (i+1)2^{i+1}$. Now we can substitute the k case from the induction hypothesis:

   $$\left(\sum_{k=1}^{i} k2^k \right) + (i+1)2^{i+1} = (i-1)2^{i+1} + 2 + (i+1)2^{i+1}$$
   $$= i \cdot 2^{i+1} - 2^{i+1} + 2 + i \cdot 2^{i+1} + 2^{i+1}$$
   $$= i(2^{i+1} + 2^{i+1}) + 2$$
   $$= i(2 \cdot 2^{i+1}) + 2$$
   $$= i2^{i+2} + 2$$

   Thus by the equations above, we have $\sum_{k=1}^{i+1} k2^k = i2^{i+2} + 2$, which is what we needed to show.

3. **Recursive definition proof [10 points]**

   Let's define a function f as follows

   $$f(0) = 1$$
   $$f(n) = f(n-1) + 4n$$

   Use induction to prove $f(n) = 2n^2 + 2n + 1$.

   [Solution]

   We will show this by induction on n.

   Base: Consider when $n = 0$: $f(0) = 1$ by the definition of f, and $2(0)^2 + 2(0) + 1 = 1$. These are equal, so the claim holds for $n = 0$.

   Induction: Suppose that the claim holds for $n = k$. That is, there is some $k \in \mathbb{N}$ such that $f(k) = 2k^2 + 2k + 1$. We need to show that $f(k+1) = 2(k+1)^2 + 2(k+1) + 1$.

Consider $f(k+1)$. By the definition of f, this equals $f(k) + 4(k+1)$. Using the induction hypothesis, we can substitute for $f(k)$:

$$\begin{aligned} f(k) + 4(k+1) &= 2k^2 + 2k + 1 + 4(k+1) \\ &= 2k^2 + 6k + 5 \\ &= (2k^2 + 4k + 2) + (2k + 2) + 1 \\ &= 2(k+1)^2 + 2(k+1) + 1 \end{aligned}$$

Thus $f(k+1) = 2(k+1)^2 + 2(k+1) + 1$, which is what we needed to show for induction.

4. **Strong induction [10 points]**

The Noble Kingdom of Frobboz has two coins: 3-cent and 7-cent.[1] Use strong induction to prove that the Frobboznics can make any amount of change ≥ 12 cents using these two coins. **You must use strong induction.**

[Solution]

The claim in the question is equivalent to the following mathematical statement:

$$\forall n \in \{i \in \mathbb{N} | i \geq 12\}, \exists t, s \in \mathbb{N}, n = 3t + 7s$$

That is, for any amount of change $n \geq 12$, there is a number of 3-cent coins t and a number of 7-cent coins s of value n. We will show that this is true by (strong) induction on n.

Base: We will need multiple base cases for our strong induction. First, consider when $n = 12$: $3(4) + 7(0) = 12$, so the claim holds with $t = 4$ and $s = 0$. It also holds for $n = 13$ with $t = 2$ and $s = 1$: $3(2) + 7(1) = 13$. Similarly, it holds for $n = 14$ with $t = 0$ and $s = 2$: $3(0) + 7(2) = 14$.

Induction: Suppose that the claim holds for n up to some k ($14 \leq n \leq k$). We need to show that $t_{k+1}, s_{k+1} \in \mathbb{N}$ exist such that $k + 1 = 3t_{k+1} + 7s_{k+1}$.

Intuitively, we'll show the $n = k + 1$ case by reaching back to the $n = k - 2$ case (similar to the postage example in lecture 18). Consider that $k + 1 = (k - 2) + 3$. Using our induction hypothesis, we can say there exist natural numbers t_{k-2} and s_{k-2} such that $k - 2 = 3t_{k-2} + 7s_{k-2}$. Substituting this in, we have:

$$\begin{aligned} k + 1 &= (k - 2) + 3 \\ &= 3t_{k-2} + 7s_{k-2} + 3 \\ &= 3(t_{k-2} + 1) + 7s_{k-2} \end{aligned}$$

Since $t_{k-2} + 1$ is a natural number, there are natural numbers $(t_{k-2} + 1)$ and s_{k-2} such that the claim holds for $n = k + 1$. This is what we needed to show for the inductive step.

5. **More fun with function composition [10 points]**

Suppose that A, B, C are sets and suppose there are functions $f : B \to C$ and $g : A \to B$. Claim: If $f \circ g$ is surjective and f is injective, then g is surjective.

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[1]They used to have a 1-cent coin, but inflation has made it essentially useless.

(a) Why did we require f to be injective? Give a concrete counter-example which shows why this condition is necessary.

[Solution]

This allows for $f \circ g$ to be onto even though g is not onto. As long as $\{f(x)|x \in Im(g)\} = C$, $f \circ g$ will be onto (i.e. every element of C is in the image of $f \circ g$). Say that g "misses" some b in its codomain. Since f is not necessarily injective, f can map b to the same element in C as some other member of B while maintaining the property from above.

For example, say that $A = \{1\}$, $B = \{pop, soda\}$, $C = \{table\}$. Define f and g as follows:

$$g(1) = pop \quad f(pop) = table$$
$$f(soda) = table$$

We see that $f \circ g(1) = table$, so $f \circ g$ is surjective. f is not injective (since $f(pop) = f(soda)$, and g is not surjective (since there is no $a \in A$ such that $g(a) = soda$).

(b) Prove the claim.

[Solution]

Here are two ways to prove it. The core construction is the same in the two proofs, but they use a different top-level outline.

Proof 1: We will prove this by contradiction. Assume that there exists f and g such that $f \circ g$ is surjective, f is injective, and g is not surjective.

Since g is not surjective, there exists $b \in B$ such that for all $a \in A, g(a) \neq b$. (Note that this is just the negation of the definiton of onto functions.) Consider one such b that is not in the image of g. b is a valid input to f, so say $f(b) = c$ for some $c \in C$. Also, since $f \circ g$ is surjective, there is $a \in A$ such that $f \circ g(a) = f(g(a)) = c$.

Say that $g(a) = b'$. Since b is not in the image of g, $b \neq b'$. But $f(g(a)) = f(b') = c = f(b)$ and f is injective. By the definition of a one-to-one function, $b = b'$. This is a contradiction. Thus, if $f \circ g$ is surjective and f is injective, g must be surjective.

Proof 2: Let b be an arbitrary element of B. We need to find an element $a \in A$ such that $g(a) = b$.

We can apply f to b, so let $c = f(b)$. Since c is an element of C and $f \circ g$ is onto, there exists an element a in A such that $(f \circ g)(a) = c$. By the definition of composition, this means that $f(g(a)) = c$.

Let $x = g(a)$. We know that $f(b) = c$ and $f(x) = f(g(a)) = c$. So $f(b) = f(x)$. Because f is one-to-one, this implies that $b = x$. But $x = g(a)$, so this means that $b = g(a)$.

So, we've found the required element a from A such that $g(a) = b$. Since this works for any element b in B, g must be onto.

You may find it useful to draw a diagram (like on pages 139 and 141 of the textbook) of the three sets and the three functions, to help develop an intuitive picture of what's happening before you try to do the proof.