CS 173, Spring 2009
Homework 4

Due *in class* on Friday, March 6th, 2009
(Total point value: 54 points.)

1. **Functions [6 points]**

   For each of the following functions, state whether or not they are one-to-one and whether or not they are onto.

   (a) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \lfloor x \rfloor$

   (b) $g : \mathbb{Z}^2 \to \mathbb{Z}$ such that $g(x, y) = xy$

   (c) $h : \mathbb{Q} \to \mathbb{R}$ such that $h(x) = x$

   Remember that $\mathbb{Z}^2$ is shorthand for $\mathbb{Z} \times \mathbb{Z}$. That is, it is the set of all pairs of integers.

2. **Reading notation [8 points]**

   An annoying but essential task in both computer science and mathematics is deciphering notation, especially when the author hasn't included enough helpful comments.

   (a) Let $f : \mathbb{P}(\mathbb{N}) \to \mathbb{N}$ such that $f(x) = |x|$. Explain in English what f does. Your explanation should include two concrete examples of input values and the corresponding output values.

   (b) Let $A = \{a, b, c, d, e\}$. List the elements of this set: $\{x : x \in \mathbb{P}(A) \text{ and } |x| = 4\}$

   (c) List the elements of this set: $\{(x, y) \in \mathbb{N}^2 \mid x + y = 3\}$

   (d) List the elements of this set: $\mathbb{Z} \cap (-3, 2]$

   Hints: In part (a), what sort of objects does f take as input? What does this tell you about the intended meaning of the vertical bars in $|x|$?

   In part (b), what type of object is x? Again, this is the key to figuring out what the vertical bar operator must mean in this expression.

   In part (d), the mismatched brackets are not a typo but, rather, a critical clue. If you've forgot what $(a, b]$ means, have a look in lecture 2.

3. **Nested quantifiers [8 points]**

   State whether the following propositions are true or false, and briefly (but clearly!) explain why.

   (a) $\exists x \in \mathbb{N}, \ \forall y \in \mathbb{N}, \text{GCD}(x, y) = 1$

   (b) $\forall x \in \mathbb{Z}, \ \exists y \in \mathbb{Z}, x = y^2$

   (c) $\forall x \in \mathbb{Q}, \ \exists (w, y) \in \mathbb{Z}^2, x = \frac{w}{y}$

   (d) $\exists (w, y) \in \mathbb{Z}^2, \ \forall x \in \mathbb{Q}, x = \frac{w}{y}$

4. **Proving a set relation [10 points]**

   (a) Prove the following set inclusion by choosing an element from the smaller set and showing that it's in the larger set.

   For any sets A, B, and C, $(A - C) - (B - C) \subseteq (A - B)$.

   (b) Show that the set inclusion doesn't hold in the other direction, by giving specific sets A, B, and C for which it fails.

   Hint: drawing yourself a Venn diagram may help you understand what's happening.

5. **Proofs with concrete functions [10 points]**

   Prove the following claims. Do this directly from the definitions of one-to-one and onto, plus high-school algebra. **Do not use calculus or theorems about increasing functions.**

   (a) Consider $f : \mathbb{N} \to \mathbb{Z}$ such that $f(x) = x^2 + 27$. Show that f is one-to-one.

   (b) Consider $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 17 - 2x$. Show that g is onto.

6. **A curious bijection [12 points]**

   Mathematicians define two sets to be the same size if you can produce a bijection from one to the other. You might imagine that the set of pairs of natural numbers is larger than the natural numbers. However, it turns out that they are the same size.

   Let's define a function f from $\mathbb{N}^2$ to $\mathbb{N}$ as follows, using a helper function s.

   - Define $s : \mathbb{N} \to \mathbb{N}$ to have the (familiar!) equation $s(n) = \sum_{i=0}^{n} i$
   - Then define $f(x, y) = s(x + y) + x$

   I claim that f is actually a bijection and so it has an inverse which is a function from $\mathbb{N}$ to $\mathbb{N}^2$.

   (a) Draw a picture of what the function does for pairs (x, y) such that $x + y \leq 4$. That is, draw a 2D graph. At the 2D position corresponding to (x, y), write the value of the function f for input (x, y). That is, at location $(1, 2)$ in your picture, write the value 7.

   (b) For some fixed natural number k, consider all pairs (x, y) with $x + y = k$. What range of output values does f produce for this set of input pairs?

   (c) What is the preimage of the output value 17?

   (d) Suppose that we have two pairs (x, y) and (p, q) which are not the same and for which $x + y \neq p + q$. Explain why $f(x, y)$ cannot be equal to $f(p, q)$.

   (e) Suppose that we have pairs (x, y) and (p, q) which are not the same but for which $x + y = p + q$. Explain why $f(x, y)$ cannot be equal to $f(p, q)$.

   Your answers to (d) and (e) could easily be combined into a proof that f is one-to-one. (But we won't ask you to actually do this.) The patterns you've seen in (a), (b), and (c) could be firmed up into a proof that f is onto.