CS 173, Spring 2009
Homework 4 Solutions

(Total point value: 54 points.)

1. **Functions [6 points]**

   For each of the following functions, state whether or not they are one-to-one and whether or not they are onto.

   (a) $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \lfloor x \rfloor$

   one-to-one: no. Note that $f(3.0) = f(3.2) = 3.0$, but $3.0 \neq 3.2$.
   onto: no. note that there is no $x \in \mathbb{R}$ such that $f(x) = 3.2$.

   (b) $g : \mathbb{Z}^2 \to \mathbb{Z}$ such that $g(x, y) = xy$

   one-to-one: no. Note that $g(1, 2) = g(2, 1) = 2$ but $(1, 2) \neq (2, 1)$.
   onto: yes. Note that for every $x \in \mathbb{Z}$, $g(x, 1) = x$.

   (c) $h : \mathbb{Q} \to \mathbb{R}$ such that $h(x) = x$

   one-to-one: yes. $h(x) = h(y)$ is equivalent to and therefore implies that $x = y$.
   onto: no. There is no $x \in \mathbb{Q}$ such that $h(x) = x = \sqrt{2}$ (other examples include $\sqrt{5}$, π, e, and all other irrational real numbers).

   Remember that $\mathbb{Z}^2$ is shorthand for $\mathbb{Z} \times \mathbb{Z}$. That is, it is the set of all pairs of integers.

2. **Reading notation [8 points]**

   An annoying but essential task in both computer science and mathematics is deciphering notation, especially when the author hasn't included enough helpful comments.

   (a) Let $f : \mathbb{P}(\mathbb{N}) \to \mathbb{N}$ such that $f(x) = |x|$. Explain in English what f does. Your explanation should include two concrete examples of input values and the corresponding output values.

   Solution: f gives the cardinality of the given subset of the natural numbers. For example, $f(\{1, 2\}) = |\{1, 2\}| = 2$, and $f(\emptyset) = |\emptyset| = 0$.

   (b) Let $A = \{a, b, c, d, e\}$. List the elements of this set: $\{x : x \in \mathbb{P}(A) \text{ and } |x| = 4\}$

   Solution: The subsets of A with cardinality 4 are given by the following set:

   $$\{\{b, c, d, e\}, \{a, c, d, e\}, \{a, b, d, e\}, \{a, b, c, e\}, \{a, b, c, d\}\}$$

   (c) List the elements of this set: $\{(x, y) \in \mathbb{N}^2 \mid x + y = 3\}$

Solution: The elements of $\mathbb{N}^2$ whose components sum to 3 are given by the set $\{(0,3),(1,2),(2,1),(3,0)\}$.

(d) List the elements of this set: $\mathbb{Z} \cap (-3, 2\,]$

Solution: The elements of the interval $(-3, 2]$ (which excludes -3 but includes 2) with integral value are given by the set $\{-2,-1,0,1,2\}$.

Hints: In part (a), what sort of objects does f take as input? What does this tell you about the intended meaning of the vertical bars in $|x|$?

In part (b), what type of object is x? Again, this is the key to figuring out what the vertical bar operator must mean in this expression.

In part (d), the mismatched brackets are not a typo but, rather, a critical clue. If you've forgot what $(a, b\,]$ means, have a look in lecture 2.

3. **Nested quantifiers [8 points]**

State whether the following propositions are true or false, and briefly (but clearly!) explain why.

(a) $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N}, \text{GCD}(x, y) = 1$

Solution: **True**. Let $x = 1$. Note that for every $y \in \mathbb{N}$, $GCD(1, y) = 1$. Notice that $GCD(1, 0)$ is also equal to 1, because 1 divides zero. GCD is not defined when both inputs are zero, but in this case we are safe because we've forced one of them to be non-zero.

(b) $\forall x \in \mathbb{Z},\ \exists y \in \mathbb{Z}, x = y^2$

Solution: **False**. Let $x = 2$. Note that there is no $y \in \mathbb{Z}$ such that $2 = y^2$.

(c) $\forall x \in \mathbb{Q}, \exists (w, y) \in \mathbb{Z}^2, x = \frac{w}{y}$

Solution: **True**. One possible definition of $\mathbb{Q}$ is $\{x \in \mathbb{R} | \exists (w, y) \in \mathbb{Z}^2, x = \frac{w}{y}\}$.

(d) $\exists (w, y) \in \mathbb{Z}^2,\ \forall x \in \mathbb{Q}, x = \frac{w}{y}$

Solution: **False**.
Consider arbitrary $(w, y) \in \mathbb{Z}^2$. If $y = 0$, the rest of the statement is clearly false. So let's suppose that $y \neq 0$. There does exist a $p \in \mathbb{Q}$ such that $p = w/y$. However, we know that $\mathbb{Q}$ contains more than one number (a lot more than just one number!). So there also exists a $q \in \mathbb{Q}$ such that $q \neq p$, i.e. $q \neq w/y$.

4. **Proving a set relation [10 points]**

   (a) Prove the following set inclusion by choosing an element from the smaller set and showing that it's in the larger set.

   For any sets A, B, and C, $(A - C) - (B - C) \subseteq (A - B)$.

   Solution:
   Suppose that $x \in (A-C)-(B-C)$. Then $x \in A-C$ and $x \notin B-C$. Because $x \in A-C$, $x \in A$ and $x \notin C$.
   Because $x \notin B - C$, then the following is false: $x \in B$ and $x \notin C$. In other words, it's true that $x \notin B$ or $x \in C$. But we already know that $x \notin C$, so this means that $x \notin B$.
   Since $x \in A$ and $x \notin B$, we have that $x \in A - B$.

   (b) Show that the set inclusion doesn't hold in the other direction, by giving specific sets A, B, and C for which it fails.

   Solution: Let $A = \{a\}$, $B = \emptyset$ and $C = \{a\}$.
   Although $a \in A - B = \{a\}$, $a \notin (A - C) - (B - C) = \emptyset - \emptyset = \emptyset$.

   Hint: drawing yourself a Venn diagram may help you understand what's happening.

5. **Proofs with concrete functions [10 points]**

   Prove the following claims. Do this directly from the definitions of one-to-one and onto, plus high-school algebra. **Do not use calculus or theorems about increasing functions.**

   (a) Consider $f : \mathbb{N} \to \mathbb{Z}$ such that $f(x) = x^2 + 27$. Show that f is one-to-one.

   Solution: Suppose that $f(x) = f(y)$ for some $x, y \in \mathbb{N}$. Then $x^2 + 27 = y^2 + 27$, hence $x^2 = y^2$, hence $x = y$ (remember that $\mathbb{N}$ does not contain the negative integers so we do not have to consider the case where $x = -y$.)

   (b) Consider $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 17 - 2x$. Show that g is onto.

   Solution: Consider some value $y \in \mathbb{R}$ in the co-domain. Let $x = \frac{17-y}{2}$. Then $x \in \mathbb{R}$ is in the domain, and $g(x) = 17 - 2\frac{17-y}{2} = 17 - (17 - y) = y$.

6. **A curious bijection [12 points]**

   Mathematicians define two sets to be the same size if you can produce a bijection from one to the other. You might imagine that the set of pairs of natural numbers is larger than the natural numbers. However, it turns out that they are the same size.

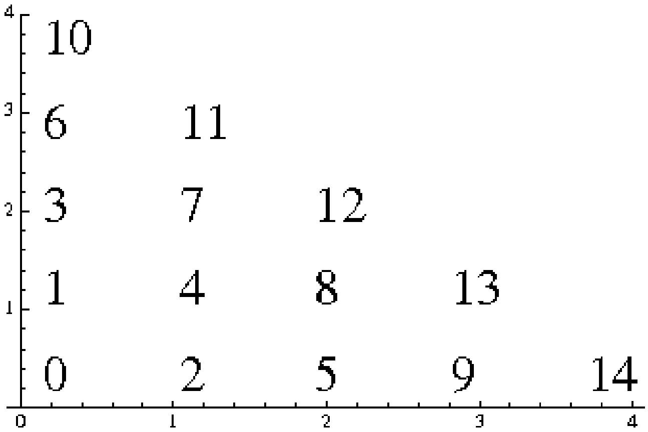
   Let's define a function f from $\mathbb{N}^2$ to $\mathbb{N}$ as follows, using a helper function s.

   - Define $s : \mathbb{N} \to \mathbb{N}$ to have the (familiar!) equation $s(n) = \sum_{i=0}^{n} i$
   - Then define $f(x, y) = s(x + y) + x$

   I claim that f is actually a bijection and so it has an inverse which is a function from $\mathbb{N}$ to $\mathbb{N}^2$.

(a) Draw a picture of what the function does for pairs (x, y) such that $x + y \leq 4$. That is, draw a 2D graph. At the 2D position corresponding to (x, y), write the value of the function f for input (x, y). That is, at location $(1, 2)$ in your picture, write the value 7.

Solution:
Your graph should look something like this:



(b) For some fixed natural number k, consider all pairs (x, y) with $x + y = k$. What range of output values does f produce for this set of input pairs?

Solution: Consider the values of x, y satisfying $x + y = k$.
Because we are in $\mathbb{N}$, for any such values of x and y we have that $y \geq 0$ and therefore $x \leq k$. For any value $x \leq k$, we can let $y = k - x$ to achieve $x + y = k$.
Thus, x ranges from 0 to k, and $f(x, y) = s(x + y) + x = s(k) + x$ ranges from $s(k)$ to $s(k) + k$. Remembering from lecture that $s(k) = \frac{k(k+1)}{2}$, we can also write this as:

$$\frac{k(k+1)}{2} \leq f(x, y) \leq \frac{k(k+1)}{2} + k$$

(c) What is the preimage of the output value 17?

Solution:
The preimage of 17 is $\{(2, 3)\}$. Note that $f(2, 3) = s(5) + 2 = 15 + 2 = 17$.
We can show that $(2, 3)$ is the only element in the pre-image by noting from our solution to part d) that, for all x, y, if $f(x, y) = f(2, 3)$, then $x + y = 2 + 3 = 5$. Testing all such values of x and y shows that $(2, 3)$ is the only element in the pre-image of 17.
It would also have been sufficient to prove that f is one-to-one, but this takes considerably more effort.

(d) Suppose that we have two pairs (x, y) and (p, q) which are not the same and for which $x + y \neq p + q$. Explain why $f(x, y)$ cannot be equal to $f(p, q)$.

Solution: Let $k = x + y$, $l = p + q$. We are assuming that $k \neq l$. So, without loss of generality, assume that $k < l$. (If k was bigger than l, we could just swap the names of the two variables.) We aim to show that $f(x, y) < f(p, q)$.

From the solution to b), we know that the sums of the coordinates k and l restrict the output values to very limited ranges. So, $f(x, y)$ has to be no bigger than the upper end of the range of outputs for the sum $k = x + y$. That is:

$$f(x, y) \leq \frac{k(k+1)}{2} + k$$

Similarly, $f(p, q)$ has to be at least as big as the lower end of the range of outputs for the sum $l = p + q$. That is:

$$\frac{l(l+1)}{2} \leq f(p, q)$$

Thus, to show that $f(x, y) < f(p, q)$, it suffices to show that $\frac{k(k+1)}{2} + k < \frac{l(l+1)}{2}$. Since $k < l$, we have that $k + 1 \leq l$ and therefore, substituting $k + 1$ for l, we have that $\frac{(k+1)(k+2)}{2} \leq \frac{l(l+1)}{2}$. It therefore suffices to show that $\frac{k(k+1)}{2} + k < \frac{(k+1)(k+2)}{2}$, which we do as follows:

$$\frac{k(k+1)}{2} + k = \frac{k(k+1) + 2k}{2} = \frac{k^2 + 3k}{2} < \frac{k^2 + 3k + 2}{2} = \frac{(k+1)(k+2)}{2}$$

(e) Suppose that we have pairs (x, y) and (p, q) which are not the same but for which $x + y = p + q$. Explain why $f(x, y)$ cannot be equal to $f(p, q)$.

Solution: Assume the contrary, that $f(x, y) = f(p, q)$. Further, let $k = x + y = p + q$. Then:

$$\begin{aligned} f(x, y) &= f(p, q) \\ s(x + y) + x &= s(p + q) + p \\ s(k) + x &= s(k) + p \\ x &= p \end{aligned}$$

Since $x = p$ and $x + y = p + q$, we have that $y = q$. But we assumed that $(x, y) \neq (p, q)$, contradiction.

Your answers to (d) and (e) could easily be combined into a proof that f is one-to-one. (But we won't ask you to actually do this.) The patterns you've seen in (a), (b), and (c) could be firmed up into a proof that f is onto.