CS 173, Spring 2009
Homework 3 Solutions

(Total point value: 50 points.)

1. **Euclidean algorithm [4 points]**

   Trace the execution of the Euclidean algorithm (lecture 9, p 229 in Rosen) on the inputs 1224 and 850. That is, give a table showing the values of the main variables (x, y, r) for each pass through the loop.
   [Solution]

x	y	r
1224	850	374
850	374	102
374	102	68
102	68	34
68	34	0
34	0	

2. **Numbers...[10 points]**

   (a) Convert $(10110111)_2$ to decimal notation.
   [Solution] $(10110111)_2 = 2^7 + 2^5 + 2^4 + 2^2 + 2^1 + 1 = 183$

   (b) Convert $(11111010110111101101)_2$ to hexadecimal notation.
   [Solution] $(11111010110111101101)_2 = (FADED)_{16}$

   (c) Convert $(CAFE8)_{16}$ to binary notation.
   [Solution] $(CAFE8)_{16} = (11001010111111101000)_2$

   (d) Convert $(1234)_{10}$ to binary notation.
   [Solution] $(1234)_{10} = (10011010010)_2$

   (e) How many bits are required to represent a positive integer n in binary notation? Express your answer as a function of n.

   [Solution] $(\lfloor \log_2(n) \rfloor) + 1$ or $\lceil \log_2(n+1) \rceil$.

   The answer $\lceil \log_2(n) \rceil$ is almost right, but it doesn't do the right thing on inputs that are powers of two.

   Since n was stipulated to be a POSITIVE integer, we don't have to worry about zero. If you did want to include zero, the answer would need to be $\max((\lfloor \log_2(n) \rfloor) + 1, 1)$.

(f) How many hexadecimal "digits" are required to represent the positive integer n? Express your answer as a function of n.
[Solution] $(\lfloor \log_{16}(n) \rfloor) + 1$ or $\lceil \log_{16}(n+1) \rceil$.

(g) Suppose you have 127 coins and 10 bags.How can you divide the coins among the bags so that you can give out any number from 1 to 127 coins without opening the bags?

[Solution] Use bags that hold powers of 2 number of coins. That is, use bags with 1, 2, 4, 8, ..., 64 coins in them. Since every number can be represented in binary, you can give any number of coins using the appropriate bags. For example 20 can be represented as $(10100)_2$ which means use bags containing 4, and 16 coins.

3. **Proof by contradiction [10 points]**

Use proof by contradiction to prove the following claim. Do this directly from the definition of the "divides" relation (section 3.4 of Rosen, lecture 7), i.e. do not use any other facts about divides that may have been proved in class or in the text.

> For all integers x and y, if $3x + 5y = 47$ then at least one of x and y is not divisible by 7.

[Solution] Suppose our claim isn't true. That is, $3x + 5y = 47$, and x and y are both divisible by 7. Then we have, $x = 7t$, and $y = 7q$. Thus, $3x + 5y = 3(7t) + 5(7q) = 7(3t + 5q)$. But, $3x + 5y = 7(3t + 5q) = 47$ which means 47 is a multiple of 7, which is a contradiction. So we must have been wrong in our assumption that the claim was false. Therefore, the claim must have been true.

4. **Set operations [12 points]**

Let's define sets as follows:

$$A = \{68, 28\}$$

$$B = \{\text{rain}, \text{snow}, \text{sun}\}$$

$$C = \{\text{water}, \text{ice}\}$$

$$D = \{\{\text{water}\}, \{\text{milk}\}\}$$

$$E = \{(\text{water}, \text{ice})\}$$

$$F = \{\text{ink}\}$$

For each of the following expressions, list the elements of the set or calculate the value (as appropriate).

(a) $\mathbb{P}(B)$

[Solution]

$\mathbb{P}(B) = \{\{\text{rain}\}, \{\text{snow}\}, \{\text{sun}\}, \{\text{rain}, \text{snow}\}, \{\text{rain}, \ \text{sun}\}, \{\text{snow}, \text{sun}\}, \{\text{rain}, \text{snow}, \text{sun}\}, \emptyset\}$

(b) $\mathbb{P}(E)$

[Solution]

$\mathbb{P}(E) = \{ \ \{(\text{water}, \text{ice})\}, \emptyset\}$

(c) $(A \times F) \cup D$

$(A \times F) = \{(68, \text{ink}), (28, \text{ink})\}$

[Solution]

$(A \times F) \cup D = \{(68, \text{ink}), (28, \text{ink}), \{\text{water}\}, \{\text{milk}\}\}$

(d) $\mathbb{P}(C) - D$

[Solution]

$P(C) = \{\{\text{water}\}, \{\text{ice}\}, \{\text{water}, \text{ice}\}, \emptyset\}$

$P(C) - D = \{ \ \{\text{ice}\}, \{\text{water}, \text{ice}\}, \emptyset\}$

(e) $\mathbb{P}(C) \cap \mathbb{P}(E)$

$P(C) = \{\{\text{water}\}, \{\text{ice}\}, \{\text{water}, \text{ice}\}, \emptyset\}$

$\mathbb{P}(E) = \{ \ \{(\text{water}, \text{ice})\}, \emptyset\}$

$\mathbb{P}(C) \cap \mathbb{P}(E) = \{\emptyset\}$

(f) $|\mathbb{P}(A \cup B) \cup \mathbb{P}(D \cup E)|$

[Solution]

$A \cup B = \{68, 28, \text{rain}, \text{snow}, \text{sun}\}$

$D \cup E = \{\{\text{water}\}, \{\text{milk}\}, (\text{water}, \text{ice})\}$

Notice that the two sets don't have any elements in common.

$\{\mathbb{P}(A \cup B) \cap \mathbb{P}(D \cup E)\} = \{\emptyset\}$

$|\mathbb{P}(A \cup B) \cap \mathbb{P}(D \cup E)| = 1$

So their powersets will only have one element in common: the empty set. This allows us to calculate the size of the union without actually writing out all the elements in it. Using the formula for the size of a power set, we have:

$|\mathbb{P}(A \cup B)| = 32$

$|\mathbb{P}(D \cup E)| = 8$

To a first approximation, we can just add these together, but we have to avoid counting the empty set twice:

$|\mathbb{P}(A \cup B) \cup \mathbb{P}(D \cup E)| = (32 + 8) - 1 = 39$

Recall that $\mathbb{P}(X)$ is the power set of X. Show your work.

5. **A Semi-Numerical Algorithm [10 points]**

The following algorithm takes as input a natural number c and a list of $n+1$ natural numbers a_0,...,a_n and yields a single natural number p as output.

procedure DoSomething(c, a_0, $\ldots$, a_n: natural numbers)
$p := a_n$
for $i := 1$ to n
    $p := (p \times c) + a_{n-i}$

(a) If $c = 10$ and the input list is $1, 2, 3, 4$ what is the output p?
[Solution]
$p := 4$(initial value at $p := a_n$)
$p := 43$(outer for loop $i := 1$)
$p := 432$(outer for loop $i := 2$)
$p := 4321$(outer for loop $i := 3$)
When the algorithm finishes, the output p is 4321.

(b) Describe in words what DoSomething does. That is, give descriptive names to the input c and the inputs a_0, $\ldots$, a_n, and then explain how the output value is related to these inputs.
[Solution]
The output value is the number: $\sum_{i=0}^{n}(a_i * (c^i))$. If the second and followingr inputs are less than c, the output is the number whose base-C representation has digits $a_n, \ldots, a_0$. (This algorithm is known as Horner's method.)

(c) How many multiplications and additions does the algorithm perform given a list of $n+1$ natural numbers a_0,...,a_n as input? Include only the operations plainly visible on the last line of the code. E.g. do not include the additions required to increment the loop index i in your answer. Express your answer as a function of n.
[Solution]
The algorithm performs n multiplications, and n additions.
The total number of multiplications and additions is thus equal to: $2n$

(d) Consider the binary expansion $b_n 2^n + b_{n-1} 2^{n-1} + ... + b_1 2^1 + b_0$. If we evaluate this expansion by performing the addition, multiplication, and exponentiation operations exactly as written, how many multiplications and additions are performed? In your answer, count each exponentiation operation as a set of multiplications (e.g. evaluating

2^3 requires 2 multiplications). Express your answer as a function of n.
[Solution]
The number of multiplications is calculated as: $(1+(n-1))+(1+(n-2))+...+1+0 = n+(n-1)+(n-2)+...+1+0 = \frac{n(n+1)}{2}$
The number of additions is equal to: n
The total number of multiplications and additions is equal to: $n + \frac{n(n+1)}{2} = \frac{n(n+3)}{2}$

6. **Fun with paradoxes [4 points]**

   In a children's book (whose identity I'm deliberately hiding), our hero is blocked by a powerful man who says

   > We will play a game to decide which way you will die. You may say one thing, and one thing only. If what you say is true, I will strangle you with my bare hands. If what you say is false, I will cut off your head."

   Conveniently, the powerful man is like many such characters in fantasy and science fiction: he self-destructs if he makes promises that turn out to contain a logical contradiction. How should our hero answer the question, so as to create a contradiction?

   [Solution] There's probably more than one solution to this question. One approach is for our hero to say 'I am a liar', or 'I don't tell the truth'. Since this can't be either true or false, neither of his opponents alternatives will apply.

   Another approach (the one used in the original source) is to say something like P = "My head will be cut off." If P is true, then the opponent's first alternative should apply, but that specifies a different mode of death, so P is false. If P is false, then the second alternative applies, but its mode of death is the same as P, so P must be true. Either way, there's a contradiction.

   This particular version of the Liar's Paradox comes from "The Lake of Tears," a Deltora Quest novel by Emily Rodda.