CS 173: Discrete Mathematical Structures, Spring 2009
Homework 11 Solutions

1. **[10 points] Proving an operation well-defined**

   Suppose that $A = \mathbb{R}^2 - \{(0,0)\}$, i.e A is 2D space minus the origin. We can define an equivalence relation $\sim$ on A as follows:

   > $(x, y) \sim (p, q)$ if and only if there is a positive real number λ such that $(x, y) = \lambda(p, q)$
   > i.e. $x = \lambda p$ and $y = \lambda q$.

   This relation treats two points as equivalent if they lie on the same ray, so each equivalence class of $\sim$ is a ray from the origin. (We saw a 3D version of this relation in lecture 36.)

   We can define addition on these rays as follows:

   $$[(x, y)] + [(p, q)] = [(xq + yp, yq - xp)]$$

   To understand where this formula comes from, think about each equivalence class as represented by the ray with unit length. The y coordinate of this ray is the sine of its angle and the x coordinate is its cosine. This addition formula is the formula for computing the sine and cosine of the sum of two angles. This formula is used in computer graphics to rotate geometric objects.

   Prove that this ray addition operation is well-defined. That is, pick two representative elements from each of the input equivalence classes and show that the corresponding outputs are in the same equivalence class.

   Solution: Let us pick two different representatives for each input: $(x, y) \sim (v, w)$ and $(p, q) \sim (r, s)$. In order to prove that the function $+$ is well defined, we need to show that the outputs we get are equivalent: $(xq + yp, yq - xp) \sim (vs + wr, ws - vr)$.

   We know that since $(x, y) \sim (v, w)$, there exists a positive real number λ_1 such that $x = \lambda_1 v$ and $y = \lambda_1 w$. Also since $(p, q) \sim (r, s)$ we must have that there exists a positive real number λ_2 such that $p = \lambda_2 r$ and $q = \lambda_2 s$. From these we have that,

   $$\begin{aligned} & (xq + yp, yq - xp) \\ = \; & (\lambda_1 v \cdot \lambda_2 s + \lambda_1 w \cdot \lambda_2 r, \lambda_1 w \cdot \lambda_2 s - \lambda_1 v \cdot \lambda_2 r) \\ = \; & (\lambda_1 \lambda_2 (vs + wr), \lambda_1 \lambda_2 (ws - vr)) \\ = \; & \lambda_1 \lambda_2 ((vs + wr), (ws - vr)) \end{aligned}$$

   Since both λ_1 and λ_2 are positive real numbers, $\lambda_1 \lambda_2$ is also a positive real number. So, $(xq + yp, yq - xp) \sim (vs + wr, ws - vr)$ by the definition of $\sim$. This completes the proof.

2. **[10 points] To infinity, and beyond!**

We have seen that it is possible to compare the size, or *cardinality*, of infinite sets. It is possible to prove that two infinite sets A and B have the same cardinality by finding a function $f : A \to B$ and proving that function is both *one-to-one* and *onto*. A set is *countably infinite* if it has the same cardinality as the positive integers.

We have also seen a proof demonstrating that set of the real numbers is in some sense larger than the set of positive integers. In 1873, Georg Cantor proved a similar fact by showing that for any set S, there is no function from S onto its power set $P(S)$. This means that for any set S, the power set $P(S)$ is always larger than S.

(a) Prove that the set of positive integers that are multiples of 10 is countably infinite.

Solution: Let us denote the set of positive integers that are multiples of 10 by A. We can prove the claim by defining a function $f : \mathbb{Z}^+ \to A$ such that $f(x) = 10x$. For any two arbitrary inputs x and y, if $f(x) = f(y)$, then $10x = 10y$ and so $x = y$. So, f is one-to-one. Also for every $a \in A$, the number $z = \frac{a}{10}$ is a positive integer because a is positive and a multiple of 10. And $f(z) = a$. So, f is also onto. So f is a one-to-one correspondence.

(b) How many sizes of infinite sets are there? In other words, how many different infinite cardinalities are there? Consider what Cantor's Theorem says about the cardinality of $P(\mathbb{N})$ and the cardinality of $P(P(\mathbb{N}))$ and...

Solution: By Cantor's theorem, the cardinality of $P(\mathbb{N})$ is larger than cardinality of $\mathbb{N}$, the cardinality of $P(P(\mathbb{N}))$ is larger than cardinality of $P(\mathbb{N})$ and so on. Similarly for the set of reals $\mathbb{R}$, the set of positive integers, even integers, odd integers, $\ldots$. So ultimately there is an infinite number of infinite cardinalities out there.