CS 173: Discrete Mathematical Structures, Spring 2009
Homework 10

Due at class on Friday, May 1, 2009 (50 points total)

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1. **[10 points] Paths and Circuits in Graphs**

   (a) Under what conditions does the graph $K_{m,n}$ have an Eulerian circuit? What has to be true about m and n?

   (b) Under what conditions does the graph Q_n have an Eulerian circuit? What has to be true about n?

   (c) Consider the complete graph K_n. Suppose we pick two vertices u and v. A path of length k between u and v is a sequence of k edges starting at u and ending at v. **Consider a path in which no vertex or edge is visted more than once.** How many different such paths of length 4 are there between u and v, assuming $n \geq 5$? Can you generalize this result and give a formula for the number of such paths of length k in K_n when $n > k$?

2. **[10 points] Graph Diameters**
   On a connected simple graph G we can measure the distance between two distinct vertices v_i and v_j as the number of edges on the shortest path between them. The *diameter* of a graph G is the maximum distance between any two distinct vertices in G.

   (a) What are the diameters of the following graphs: K_n, C_n, and W_n?

   (b) Prove by induction that the diameter of the n-dimensonal hypercube Q_n is n.

3. **[10 points] Properties of Relations**

   (a) The relation E relates intervals of the real line that abut one another. Specifically $(x, y)E(p, q)$ if and only if $y = p$ or $x = q$. E.g. $(2, 3)$ and $(1.5, 2)$ are related because they share the common endpoint 2. Using a specific concrete counter-example, prove that this is not an equivalence relation.

   (b) Suppose that Q is the relation on positive real numbers such that xQy if and only if $xy = 1$. Is Q reflexive, irreflexive, both, or neither? Is Q transitive? Briefly justify your answers.

   (c) Define the relation T on the set $\mathbb{N}^3$ by saying that $(x, y, z)T(p, q, r)$ if and only if $x+y+z = p+q+r$. List three elements of $[(1, 2, 3)]$ and also one element of $\mathbb{N}^3$ that is not in $[(1, 2, 3)]$.

4. **[10 points] Proving relation properties**

   (a) Let $\ll$ be the relation on $\mathbb{Z}^2$ such that $(x, y) \ll (p, q)$ if and only if either $x < p$, or else $x = p$ and $y \leq q$. That is, when the first coordinates are different, they determine the ordering of pairs, e.g. $(0, 8) \ll (1, 3)$. But when the first coordinates are the same, we compare the second coordinates, e.g. $(1, 3) \ll (1, 8)$. Prove that $\ll$ is antisymmetric.

   (b) Let $\sim$ be the relation on $\mathbb{Z}$ such that $x \sim y$ if and only if $4 \mid 3x + 5y$. Prove that $\sim$ is transitive.

5. **[10 points] A Probabilistic Algorithm**

In the last homework, we saw an algorithm to verify polynomial identities based on the binomial theorem. In this problem, we will consider a probabilistic algorithm to verify polynomial identities of the form:

$$(a_1 x + a_2)^n = b_0 x^n + b_1 x^{n-1} + \ldots + b_{n-1} x^1 + b_n$$

where n is a positive integer and the a_i and b_i are non-negative integers. We will refer to the left-hand side of the identity as $G(x)$ and the right-hand side as $F(x)$, so we have $G(x) = (a_1 x + a_2)^n$ and $F(x) = b_0 x^n + b_1 x^{n-1} + \ldots + b_{n-1} x^1 + b_n$

One way to verify the identity is to use an algorithm to test each coefficient generated by $G(x)$ and make sure it matches the corresponding coefficient in $F(x)$. This is similar to the algorithm on the last homework assignment and would require $\Theta(n^2)$ operations.

Another option would be to randomly pick a value for x and verify that the two sides of the equation yield the same answer. This would be a kind of probabilistic algorithm, in that it would yield the right answer when $G(x) = F(x)$ but not always give us the right answer when $G(x) \neq F(x)$. In analyzing this algorithm we need to consider both how many operations it will perform and the probability that it will give us an incorrect answer. Here is the pseudo-code for the algorithm:

procedure ProbablyVerify(x,a_1, a_2,n,b_0..., b_n)
$binomial := (a_1 x) + a_2$
$g := binomial$
for $i := 2$ to n
begin
  $g := g \cdot binomial$
end
$f := 0$
$xpower := 1$
for $j := 0$ to n
begin
  $f := f + (b_{n-j} \cdot xpower)$
  $xpower := xpower \cdot x$
end
if $(g = f)$ **then**
  $matches :=$ **true**
else
  $matches :=$ **false**
return $matches$

(a) State a big-theta bound on the number of operations done by the procedure **ProbablyVerify** in terms of the degree of the polynomial which is given by the input n .

(b) The procedure will give an incorrect answer when we choose a specific value $x = c$ such that $G(c) = F(c)$, but it is **not true** that for all real numbers x that $G(x) = F(x)$. This happens when we accidentally choose a value c such that $G(c) - F(c) = 0$. In other words, we chose a value c that is a root of the polynomial equation $G(x) - F(x) = 0$. A degree n polynomial has at most n distinct roots. Given that fact, if we choose an integer x uniformly at random from the range 0 to m, for what value of m is the probability of selecting a root definitely at or below 0.01? Explain your answer.