CS 173, Spring 2009
Homework 0 Solutions

1. **Logarithms, exponents, complex numbers [6 points]**
Simplify the following expressions as much as possible, **without using a calculator (either hardware or software)**. Do not approximate. Express all rational numbers as fractions. For complex numbers use i to represent the value $\sqrt{-1}$.

[Solutions inline]

(a) $\sqrt[3]{\sqrt{\pi^{1200}}} = \sqrt[3]{\pi^{600}} = \pi^{200}$

(b) $\frac{(2^9 \times 2^7)^4}{256} = \frac{(2^{16})^4}{2^8} = \frac{2^{64}}{2^8} = 2^{56}$

(c) $\log_5 625^n = n(\log_5 625) = 4n$

(d) $(\log_3 63) - (\log_3 7) = \log_3 \frac{63}{7} = \log_3 9 = 2$

(e) $\frac{\log_{10} 4096}{\log_{10} 2} = \log_2 4096 = 12$

(f) $(1-i)(2-i)(3-i) = (2-3i+i^2)(3-i) = (2-3i-1)(3-i) = (1-3i)(3-i)$

$= (3-10i+3i^2) = -10i$

2. **Numbers of all kinds...[4 points]**
In mathematical writing it is customary to denote certain common sets of numbers using special symbols (often a single letter). For example, the set of rational numbers is typically denoted as **Q**. We will follow the textbook conventions, listed on the inside cover of your textbook, for assigning symbols to these sets.

(a) What letter is used to denote the set of numbers $\{..., -2, -1, 0, 1, 2, ...\}$? What is this set called?
[Solution]
The integers, represented by **Z**.

(b) How is the set of numbers $\{1, 2, ...\}$ denoted and what is this set called?
[Solution]
The positive integers, represented by $\mathbf{Z}^+$.

(c) (2 points) What is the formula for computing $\frac{a+bi}{b+ci}$ where $i = \sqrt{-1}$?
Hint: look it up on the web or in a reference book.
[Solution]
The standard formula for computing $\frac{a+bi}{c+di}$ is $\frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$. To apply this to our fraction, we need to replace c with b and d with c. This gives us the answer $\frac{ab+bc}{b^2+c^2} + \frac{b^2-ac}{b^2+c^2}i$. You can also derive this answer by multiplying the original fraction by $\frac{b-ci}{b-ci}$.

3. **Floors and ceilings [5 points]**

   (a) What is $\lceil \lfloor \frac{1}{2} \rfloor + \lfloor \frac{1}{3} \rfloor - \frac{1}{3} + \frac{1}{2} \rceil$?
   [Solution]
   $= \lceil 0 + 0 + \frac{1}{6} \rceil = 1$

   (b) What is $\lfloor -63.5 \rfloor$?
   [Solution]
   $= -64$

   (c) What is $\lfloor 5.5 \rfloor - \lceil -5.5 \rceil$
   [Solution]
   $= 5 - (-5) = 10$

   (d) Is $\lceil 2x \rceil \leq \lceil x \rceil + \lceil x - \frac{1}{2} \rceil$ for all real numbers x? Just answer yes or no, you don't need to justify your answer.
   [Solution]
   Yes.

   (e) Is $\lceil x - y \rceil = \lceil x \rceil - \lceil y \rceil$ for all real numbers x and y? Justify your answer.
   [Solution]
   No, consider $x = .2$ and $y = .1$: $\lceil .2 - .1 \rceil = 1 \neq \lceil .2 \rceil - \lceil .1 \rceil = 0$

4. **Functions [5 points]**
   Suppose $F(x) = x^2 - 4x$ and $G(x) = x + 4$ and $H(x) = x^2 - 4$.

   (a) What is $F(G(z))$?
   [Solution]
   $F(G(z)) = F(z+4) = (z+4)^2 - 4(z+4) = z^2 + 4z$

   (b) What is $F(G(G(G(G(G(-20))))))$?
   [Solution]

$$\begin{aligned} F(G(G(G(G(G(-20)))))) &= F(G(G(G(G((-20)+4))))) = F(G(G(G((-16)+4)))) \\ &= F(G(G((-12)+4))) = F(G((-8)+4)) \\ &= F((-4)+4) = (0)^2 - 4(0) = 0 \end{aligned}$$

   (c) Express $F(x) + H(G(y))$ as a single function.
   [Solution]
   $I(x,y) = F(x) + H(G(y)) = x^2 - 4x + (y+4)^2 - 4 = x^2 - 4x + y^2 + 8y + 12$

   (d) Simplify $\frac{G(F(x))}{H(x)}$ as much as possible.
   [Solution]
   Factor both the numerator and the denominator:
   $\frac{G(F(x))}{H(x)} = \frac{(x^2-4x)+4}{x^2-4} = \frac{(x-2)(x-2)}{(x-2)(x+2)} = \frac{(x-2)}{(x+2)}$, undefined when $x = 2$ or $x = -2$.

5. **Sums and products [10 points]** Please show your work in deriving the solutions to the following questions. Note that in these questions, i is an index variable and is not representing an imaginary number.

   (a) Express $S(n) = \sum_{i=0}^{n-1}(2i+1)$ as a simple function of the variable n.

   [Solution]
   $$\sum_{i=0}^{n-1}(2i+1) = 2\sum_{i=0}^{n-1}(i) + \sum_{i=0}^{n-1}(1) = 2\frac{(n)(n-1)}{2} + (n) = (n)(n-1) + n = n^2$$

   (b) Express $P(n) = \prod_{i=1}^{n} \frac{3(i+1)}{3i}$ as a simple function of the variable n.

   [Solution]
   $$\prod_{i=1}^{n} \frac{3(i+1)}{3i} = \prod_{i=1}^{n} \frac{(i+1)}{i} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n+1}{n} = \frac{2 \cdot 3 \cdot 4 \cdots (n+1)}{1 \cdot 2 \cdot 3 \cdots n} = n+1$$

   (c) Rewrite $\sum_{i=1}^{n} \frac{2^i}{i^2}$ as a sum which has an index from 0 to $n-1$.

   [Solution]
   $$\sum_{i=1}^{n} \frac{2^i}{i^2} = \sum_{i=0}^{n-1} \frac{2^{(i+1)}}{(i+1)^2}$$

   (d) Express $S(n) = \sum_{i=0}^{n}(4i+1)$ as a simple function of the variable n.

   [Solution]
   $$\sum_{i=0}^{n}(4i+1) = 4\sum_{i=0}^{n}(i) + \sum_{i=0}^{n}(1) = 4\frac{n(n+1)}{2} + (n+1) = 2n^2 + 3n + 1$$

   (e) Express $S(n) = \sum_{i=1}^{n} \frac{1}{i^2(i^2+1)}$ as a simple function of the variable n.

   hint: can you rewrite $\frac{1}{i^2(i^2+1)}$ *as a difference of two fractions?*

   [Omitted]

6. **Algorithms [10 points]**
The word *algorithm*, derived from the name of the ninth-century Persian mathematician al-Khowārizmī, refers to a step-by-step method for solving a problem. We will use *pseudo-code* to describe algorithms in this class. This should save you time, since you don't have to learn another full programming language. It also will make the algorithm descriptions more concise and understandable. Our version of pseudo-code resembles a very minimal imperative programming language and is described in Appendix 3 of the textbook. When reading an algorithm description, if anything about the pseudocode convention is unclear you should consult the textbook.

The following algorithm takes as input an arbitrary list of n real numbers a_1,...,a_n and yields a single real number p as output.

procedure DoSomething(a_1,..., a_n)
$p := a_1$
for $i := 2$ to n
    $temp := p$
    for $j := 1$ to a_i
        $p := p \times temp$

(a) If the initial list is $10, 2, 2, 2$ what is the output p?
[Solution]
p is assigned the following values during execution of the algorithm
p $:= 10$ (initial value at $p := a_1$)
p $:= 100$, 1000 (outer for loop $i := 2$)
p $:= 10^6$, 10^9 (outer for loop $i := 3$)
p $:= 10^{18}$, 10^{27} (outer for loop $i := 4$)
When the algorithm finishes, the output p is 10^{27}.

(b) How many multiplications does the algorithm perform given a list of n real numbers a_1,...,a_n as input?
[Solution]
We have two separate solutions: one if you assumed that a_2,...,a_n are positive integers (as suggested on the newsgroup) and one if you assumed that a_2,...,a_n are real numbers (as was stated in the problem). Both solutions are below for you to see, and it's fine if you used one or the other in your submissions.
If we assume that a_2,...,a_n are positive integers, then the algorithm performs performs a_i multiplications for every $i \geq 2$. If $M(a_1, ..., a_n)$ is the number of multiplications, then:

$$M(a_1, ..., a_n) = \sum_{i=2}^{n} a_i$$

The situation is more complicated if we treat a_2,...,a_n as reals. In the pseudocode, the loop "**for** $j := 1$ to a_i" will not execute if $a_i < 1$. This means that we should ignore all a_i that are less than 1 in our summation. Even if $a_i \geq 1$, then the loop will only run

$\lfloor a_i \rfloor$ times. We can define a function $f(x)$ for a real number x that will choose the right values for us:

$$f(x) = \begin{cases} \lfloor x \rfloor & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

$f(a_i)$ floors the a_i that we want to sum and is 0 for the a_i that we want to ignore. We can now use f in our summation:

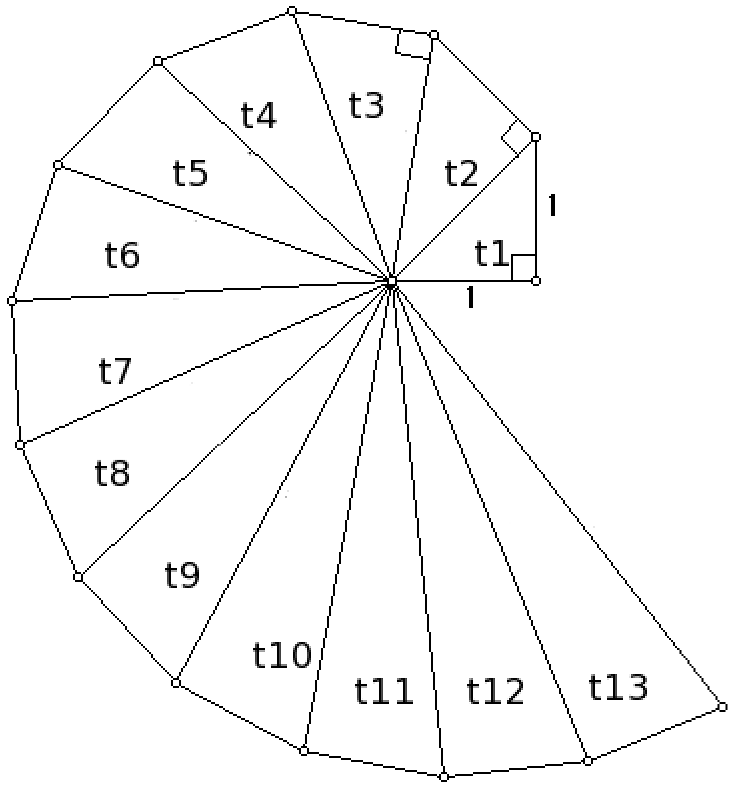
$$M(a_1, ..., a_n) = \sum_{i=2}^{n} f(a_i)$$

(c) If the input is a list of 3 real numbers a_1,a_2,a_3 express the output of the algorithm as mathematical function of the list elements.

[Solution]

$p = (a_1^{a_2+1})^{a_3+1}$ where a_2, a_3 are considered to be positive integers. If a_2, a_3 are considered to be reals, then $p = (a_1^{f(a_2)+1})^{f(a_3)+1}$ where f is defined as in the previous question.

7. **Spirals [10 points]**
Consider the spiral shown in the figure below. This construction starts with a right-angled triangle which has sides of length 1. Successive right-angled triangles are added on with a base of length 1 and a side with length equal to the hypotenuse of the previous triangle.



(a) If we denote the first triangle as t_1, how long is the hypotenuse of the nth triangle t_n?
[Solution] $hyp_n = \sqrt{n+1}$

(b) Write a formula for the total area of the first n triangles using summation notation.
[Solution] $A = \frac{1}{2}(1 \cdot 1) + \frac{1}{2}(1 \cdot \sqrt{2}) + \frac{1}{2}(1 \cdot \sqrt{(3)}) + \cdots + \frac{1}{2}(1 \cdot \sqrt{n}) = \frac{1}{2}\sum_{i=1}^{n} \sqrt{i}$

(c) ***5 point bonus question:*** For $n > 1$, do you think the total area of the first n triangles can be exactly represented by a digital computer? Briefly and clearly explain why or why not in five or fewer sentences.

[Solution] There are multiple acceptable answers to this question. A digital computer cannot represent the exact area as a number since this would require manipulating and storing irrational numbers (e.g. the area of the first 2 triangles). Digital computers use finite representations and would need infinite memory to express every digit of an irrational number. Typically, modern computer implementations will approximate such numbers using *floating point* or a similar representation. Still, it is possible for a computer to represent the area as an unevaluated algebraic expression (e.g. store the terms of the summation above: ".5*1 + .5*sqrt(2)"), which will preserve the exact value of the area in memory.