CS 173: Discrete Structures, Spring 2009
Quiz 2 Solutions

1. (7 points) Mark the following claims as "true" or "false"

   (a) $n^2 + \log_{10} n = \Theta(n^2)$

   Solution: True. We can ignore the $\log_{10} n$ term because it is dominated by the n^2 term.

   (b) $n = \Theta(n^3)$

   Solution: False. n grows slower than n^3, so $n = O(n^3)$ but not $n = \Omega(n^3)$.

   (c) If a function $f(n) = O(g(n))$ then the function $g(n) = \Omega(f(n))$

   Solution: True. This is how we defined Ω in lecture. O and Ω are opposite inequalities.

   (d) If a function $f(n) = O(h(n))$ and a function $g(n) = O(h(n))$ then the function $f(n)g(n) = O(h(n))$

   Solution: False. Suppose that $f(n) = g(n) = h(n) = n$. Then the first two relations hold, but the second is false, because $n \cdot n = n^2$ is not $O(n)$.

   (e) Let $f : \mathbb{R} \to \mathbb{Z}$ such that $f(x) = \lfloor x \rfloor$. f is onto.

   Solution: True. Since the reals include all integers and each integer maps onto itself, the output of f includes all the integers.

   (f) A function from $\mathbb{N}^2$ to $\mathbb{N}$ cannot be one-to-one because $\mathbb{N}^2$ has more elements than $\mathbb{N}$. (Remember that $\mathbb{N}^2$ is the set of all pairs of natural numbers.)

   Solution: False. We saw a one-to-one function of this sort on Homework 4.

   (g) Suppose that A and B are sets. If I prove that every element of A is also an element of B, I can conclude that $A = B$.

   Solution: False. You can only conclude that $A \subseteq B$. To show that $A = B$, we would also have to show that every element of B is also an element of A.

2. (4 points)

   One of these two statements is true and one is false. State which one is false and explain clearly why it is false.

   (a) For every integer x, there is an integer y such that $y > 3x - 14$.

   (b) There is an integer x, such that for every integer y, $y > 3x - 14$.

   Solution: The second statement (b) is false. Suppose we choose some integer x. Then the integer $y = 3x - 357$ (for example) is less than $3x - 14$

3. (4 points) Define the function $f : \mathbb{Z}^2 \to \mathbb{R}$ by $f(x, y) = x + y$. Show that f is not one-to-one by giving a specific counter-example.

   Solution: Consider $(0, 3)$ and $(3, 0)$. $f(0, 3) = 3 = (3, 0)$. So these two input points map onto the same output number, which is inconsistent with f being one-to-one.

4. (4 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function whose inputs and outputs are real numbers. Define what it means for f to be strictly increasing.

   Solution: f is strictly increasing if, for all real numbers x and y, $x < y$ implies that $f(x) < f(y)$.

5. (6 points)

Let's define a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ as follows:

- Base cases: $f(1) = 5$ and $f(2) = 10$
- Induction: $f(n) = 2f(n-1) + f(n-2)$ for all $n \geq 3$

Supply the missing (boxed) parts of the following proof by induction that $f(n) < 3^n$ for all integers $n \geq 3$.

Proof: By induction on n.

Base case or cases:

Solution: $f(3) = 2f(2) + f(1) = 25$ which is smaller than $3^3 = 27$. $f(4) = 2f(3) + f(2) = 60$ which is smaller than $3^4 = 81$.

Notice that the claim is only true for $n \geq 3$, so this should be your first base case. The second base case is required because the inductive step needs to reach back two integers (e.g. from $k+1$ back to $k-1$).

Inductive hypothesis: [Spell out the specifics of the hypothesis for the inductive step. Don't just refer to "the claim."]

Solution: Suppose that our claim holds for every integer between 3 and k. That is, for every integer j, $3 \leq j \leq k$, $f(j) < 3^j$.

Notice that our proposition $P(n)$ is $f(n) < 3^n$. We need a "strong" inductive hypothesis because the rest of the inductive step uses not only the result for k but also the result for $k-1$. The "weak" version would be that there is some $k \geq 3$ (or ≥ 4), such that $f(k) < 3^k$.

Finally, notice that our inductive step was proving the result true for $k+1$ (not k, not $n+1$), so our inductive hypothesis needs to cover values up through k (not up through $k-1$, not up through n).

This question was deliberately hard but with much potential for partial credit. So typical scores were around 3 points out of 6.

We need to show that our claim holds for $k+1$. We can assume that $(k+1) \geq 5$, since smaller values were covered by the base case(s).

By the definition of f, we know that $f(k+1) = 2f(k) + f(k-1)$.

Applying the induction hypothesis twice, we find that

$$2f(k) + f(k-1) < 2 \cdot 3^k + f(k-1) < 2 \cdot 3^k + 3^{k-1}$$

But $2 \cdot 3^k + 3^{k-1} < 2 \cdot 3^k + 3^k = 3 \cdot 3^k = 3^{k+1}$ by high-school algebra.

Combining these inequalities, we find that $f(k+1) < 3^{k+1}$, which is what we needed to show. $\square$