CS 173: Discrete Structures, Fall 2008
Quiz 2 answers

1. (1 point) Give the day and time when your assigned discussion section meets. (If you've recently switched sections, give either one.)

   [Your answer to this one will obviously vary.]

2. (5 points) Mark each of the statements as "true" or "false".

   (a) n^2 is $\Omega(n \log n)$ **True**

   (b) $87 \log n$ is $O(n^3)$ **True**

   (c) $n!$ is $O(100n^3 + 35)$ **False**

   (d) $2^k + 2^{10}$ is $O(2^k)$ **True** Notice that 2^{10} is a constant, so it doesn't affect how fast the output grows as a function of k.

   (e) The rational numbers have the same cardinality as the integers. **True**

3. (4 points) Negate the following statement, moving the "not" onto individual (non-complex) propositions.

   For every $p \in \mathbb{Z}$, there is a $q \in \mathbb{Z}$, if $p > q$, then $q^3 + p!$ is a multiple of 7.

   Solution: There is a $p \in \mathbb{Z}$, such that for every $q \in \mathbb{Z}$, $p > q$ and $q^3 + p!$ is not a multiple of 7.

4. (9 points) Each of the following definitions or claims has a significant flaw. Explain briefly what's wrong with each:

   (a) If $f : \mathbb{N} \to \mathbb{N}$ is an increasing function, then f is one-to-one.

   Solution: Saying that a function is increasing merely requires that the outputs not get smaller as the input is made larger. so $f(x) = 3$ is an increasing function. To force the function to be one-to-one, it needs to be **strictly** increasing.

   (b) If $f : A \to B$ and $g : B \to C$ are functions, then $f \circ g : A \to C$ is defined by $(f \circ g)(x) = f(g(x))$.

   Solution: Taking $f(g(x))$ only works if the codomain of g is the same set as (or at least a subset of) the domain of f. But the way we've set up f and g, the codomain of g is C but the domain of f is A. To fix this, either we'd have to compose f and g in the other order or change how we set up their domains and codomains.

   (c) Suppose that f and g are functions whose domain and range are the real numbers. Then f is $O(g)$ if there is a constant real number m such that $|f(x)| \leq m\,|g(x)|$ for every $x \geq k$.

   Solution: That variable k appears out of the blue. What is it? What are the constraints on its value? If you look back at the definition in the book, we should have said there exist two constants, m and k.

5. (6 points) Use mathematical induction to prove that

$$\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$$

for any integer $n \geq 1$. Notice that the summation starts at 1. So the first term in the summation is $\frac{1}{2}$. You must do this problem by induction, not using some other proof technique.

Solution:

> Proof by induction on n.
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> Base: If $n = 1$, the $\sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2}$ and $1 - \frac{1}{2^n}$ is also $\frac{1}{2}$. So the equation holds.
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> Induction: Suppose that $\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$. for some integer $n \geq 1$. We need to show that $\sum_{k=1}^{n+1} \frac{1}{2^k} = 1 - \frac{1}{2^{n+1}}$.
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> From the definition of summation, we know that $\sum_{k=1}^{n+1} \frac{1}{2^k} = (\sum_{k=1}^{n} \frac{1}{2^k}) + \frac{1}{2^{n+1}}$. By the inductive hypothesis, $\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$. Substituting into the previous equation, we get $\sum_{k=1}^{n+1} \frac{1}{2^k} = (1 - \frac{1}{2^n}) + \frac{1}{2^{n+1}}$.
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> But $(1 - \frac{1}{2^n}) + \frac{1}{2^{n+1}} = 1 - (\frac{1}{2^n} - \frac{1}{2^{n+1}}) = 1 - \frac{1}{2^n}(1 - \frac{1}{2}) = 1 - \frac{1}{2^n}(\frac{1}{2}) = 1 - \frac{1}{2^{n+1}}$.
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> So, by combining these two equations, we have $\sum_{k=1}^{n+1} \frac{1}{2^k} = 1 - \frac{1}{2^{n+1}}$, which is what we needed to show.

It would also be fine to start your inductive step as follows: "Suppose that $\sum_{k=1}^{n} \frac{1}{2^p} = 1 - \frac{1}{2^p}$. for all integers $p \leq n$." (I.e. set the proof up as "strong" induction.)