CS 173, Spring 2009
Midterm 1 Solutions

Problem 1: Short answer (12 points)

The following require only short answers. Justification/work is not required, but may increase partial credit if your short answer is wrong.

(a) If an if/then statement P is true, is the contrapositive of P always true? **Solution:** Yes.

(b) Let $f : \mathbb{R} \to \mathbb{Z}$ be defined by $f(x) = \lfloor x^2 \rfloor$. Name (or clearly describe) the set that is the co-domain of f.

Solution: $\mathbb{Z}$. The co-domain is the declared set of possible output values, which may be rather larger than the actual set of values the function produces.

(c) What is the base-10 equivalent of the hexidecimal number 2F?

Solution: $2 \times 16 + 15 = 32 + 15 = 47$

(d) Give a closed form expression for $\sum_{k=1}^{p} \frac{1}{2^k}$.

Solution: $1 - \frac{1}{2^p}$.

(e) If x is a real number and $\lfloor x \rfloor = \lceil x \rceil$ then what special property must x have?

Solution: x must be an integer.

(f) Suppose A is a set. Is it always the case that the empty set is a member of A?

Solution: No. The empty set is a **subset** of every set A, but that's not the same thing as being a member. The empty set is a member of A only if the definition of A specifically includes it.

Problem 2: Set theory (10 points)

(a) (2 points) How many elements are in the set $\{\{3\}, \{4, 5\}, \emptyset\}$?

Solution: Three. $\{3\}$, plus $\{4, 5\}$, and lastly $\emptyset$.

Compute the output of each of the following set operations. Recall that $\mathbb{P}(A)$ is the power set of A.

(b) (3 points) $\mathbb{P}(\{a, b, c\}) =$

Solution: $\{\{a, b, c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \emptyset\}$

(c) (3 points) $\{4, 7\} \times \{7, 9, 2\} =$

Solution: $\{(4, 7), (4, 9), (4, 2), (7, 7), (7, 9), (7, 2)\}$

(d) (2 points) $\{2, 8, 4\} \times \emptyset =$

Solution: $\emptyset$

Problem 3: Longer answers (6 points)

(a) Trace the execution of the Euclidean algorithm as it computes the GCD of 245 and 280.

Solution:

x	y	r
245	280	245
280	245	35
245	35	0
35	0	[done]

So the GCD is 35.

(b) Suppose that A and B are sets. Prove that $\mathbb{P}(A \cup B)$ is not always equal to $\mathbb{P}(A) \cup \mathbb{P}(B)$ by giving a specific counter-example and explaining briefly why it is a counter-example.

Solution: Suppose that $A = \{a\}$ and $B = \{b\}$. Then $\{a, b\}$ is in $\mathbb{P}(A \cup B)$ but not in $\mathbb{P}(A) \cup \mathbb{P}(B)$.

Problem 4: Remembering definitions (8 points)

(a) If d is a positive integer, the quotient and remainder modulo d are defined by a theorem named the "division algorithm." Finish the following statement of this theorem:

> Suppose that a is an integer and d is a positive integer, then there are unique integers q and r such that ...

Solution: $a = dq + r$ and $0 \leq r < d$.

(b) Suppose that m and n are integers, not both zero. Define what it means for an integer d to be the GCD of m and n. Assume that we already know what $a|b$ means and base your answer on this "divides" relation. Do not use prime factorizations.

Solution: (a) $d|m$, (b) $d|n$, and (c) d is the largest integer satisfying both (a) and (b).

Problem 5: Writing a proof (6 points)

Recall the following definition: Given any positive integer m, the integers a and b are *congruent modulo* m if and only if there is an integer k such that $a = b + km$. a is congruent to b modulo m is written as $a \equiv b \pmod{m}$.

Prove that, for any integers p, q, and r and any positive integer m,

> If $p \equiv q \pmod{m}$ and $q \equiv r \pmod{m}$, then $p \equiv r \pmod{m}$.

Prove this directly using the above definition, together with high school algebra. Do not use other facts about modular arithmetic proved in class or in the book.

Solution: Let p, q, and r be integers and let m be a positive integer. Suppose that $p \equiv q \pmod{m}$ and $q \equiv r \pmod{m}$.

By the definition of congruence mod m, we know that there are integers k and j such that $p = q + km$ and $q = r + jm$.

Substituting the second equation into the first, we get that $p = r + jm + km$. That is, $p = r + (j + k)m$.

Since j and k are integers, $j + k$ is an integer. So this means that $p \equiv r \pmod{m}$.

Problem 6: Logic and proof structure (8 points)

1. (3 points) Give the negation of the following statement, moving the negatives (e.g. "not") so that they are on individual predicates (e.g. "my bike has two wheels"). Although you can use shorthand to work out your answer, your final answer must be written out in words. Everything after the word "then" is intended to be read as belonging to the conclusion of the if/then statement.

   > For every book B, if B is a fantasy novel, then B must involve magic and B must feature a naive young hero.

   Solution: There is a book B such that B is a fantasy novel but either B doesn't involve magic or B does not feature a naive young hero.

2. (5 points) Prove the following claim using proof by contradiction. You **must** use proof by contradiction.

   > For any natural numbers p and q, if $pq - 17 < 80$ then either $p < 10$ or $q < 10$.

   Solution: Suppose not. That is, suppose that there are natural numbers p and q such that $pq - 17 < 80$, $p \geq 10$, and $q \geq 10$. [This is the negation of our original claim.]

   Since $p \geq 10$ and $q \geq 10$, we know that $pq \geq 100$. So $pq - 17 \geq 83$. But this is inconsistent with the statement that $pq - 17 < 80$. So we have a contradiction.

   Since our assumption that the claim was false led to a contradiction, the claim must have been true.