CS 173, Spring 2008
Final Exam, 18 December 2008

Fill in your name, netid, and discussion section time below. Also write your netid on the other pages (in case they get separated).

NAME:

NETID: | **DISC:**

Problem	1	2	3	4	5	6	7	8	9	10
Possible	8	12	10	12	10	8	10	10	10	10
Score										

Total out of 100 points

INSTRUCTIONS (read carefully)

- There are 10 problems, each on a single page. Make sure you have a complete exam.
- The point value of each problem is indicated next to the problem.
- It is wise to skim all problems and point values first, to best plan your time.
- Points may be deducted for solutions which are correct but excessively complicated, hard to understand, or poorly explained.
- Brief explanations and/or showing work may increase partial credit for buggy answers.
- We expect most people to finish the exam in 2 hours, but you can take up to the full 3 hours.
- Turn in your exam at the front. Show your ID to the proctors.
- This is a closed book exam. No notes or electronic devices of any kind are allowed.
- Do all work in the space provided, using the backs of sheets if necessary. See the proctor if you need more paper.
- Please bring any apparent bugs to the attention of the proctors.
- After the exam is over, discuss its contents with other students **only** after verifying that they have also taken the exam (e.g. they aren't about to take a conflict exam).

Problem 1: From the dawn of time (8 points)

(a) Give the negation of the following statement, moving the not's onto individual propositions. Notice that everything after "then" is intended to be part of the conclusion of the if/then statement.

> For every car m, there is exists a road r, such that if m has driven down r, then m got stuck or m had a flat tire.

(b) Suppose $A = \{1, 5\}$ and $B = \{a, b\}$. List the elements of $A \times \mathbb{P}(B)$. (Remember that $\mathbb{P}(M)$ is the power set of M.)

Problem 2: Counting and probability (12 points)

(a) How many positive integers k in the range $0 < k \leq 100$ are multiples of 7 or 5 (or both)?

(b) The number of ways to pick a k-element subset from a set containing n elements is written $C(n, k)$ or $\binom{n}{k}$. If n is even, what value of k gives us the largest value for $C(n, k)$?

(c) How many bitstrings of length 6 can be formed using four 1s and two 0s ?

Problem 3: Longer probability (10 points)

(a) The game of *chuck-a-luck* is played by rolling three dice. The gambler bets on one of the numbers 1 through 6. His chosen number may appear on zero, one, two, or three of the dice. If the number appears i times, the gambler wins i dollars. The dice act independently, so we think of each die as a Bernoulli trial with success corresponding to the gambler's number turning up. The dice are fair, so the probability of any single die rolling the gambler's number is $1/6$. Write down a formula for the probability of winning i dollars assuming $0 < i \leq 3$.

(b) Suppose a robot is located on a two-dimensional grid where its location is given by 2 real-valued numbers (x, y). The starting location is $(0, 0)$. How the robot moves depends on a coin flip: Heads means it moves 3 units in the x direction and Tails means it moves 2 in the y direction. For example, if the first three flips are Tails, Heads, Heads then the robot's position will be $(6, 2)$. Assuming Heads and Tails are equally likely, what is the robot's *expected* position after three flips? **Show your work.**

Problem 4: Graph theory (12 points)

(a) State Euler's formula relating the number of edges (e), the number of vertices (v), and the number of regions (f) in a planar graph.

(b) For a tree with n vertices, what is the sum of the degrees of all the vertices? Show your work.

(c) What is the chromatic number of the graph W_n? If the chromatic number depends on n, describe how it depends on n. Briefly explain your answer.

(d) Does the graph $K_{3,3}$ have an Eulerian circuit? Why or why not?

Problem 5: Relations and Functions (10 points)

Mark the appropriate boxes

(a) $f : \mathbb{R} \to \mathbb{Z}$ such that $f(x) = \lfloor x \rfloor$

One-to-One: ☐ **Not One-to-One**: ☐

Onto: ☐ **Not Onto**: ☐

(b) $g : \mathbb{N} \to \mathbb{Z}$ such that $f(x) = |x|$

One-to-One: ☐ **Not One-to-One**: ☐

Onto: ☐ **Not Onto**: ☐

(c) $p : \mathbb{Z} \times \{0, 1\} \to \mathbb{Z}$ such that $f(x, y) = 2x + y$

One-to-One: ☐ **Not One-to-One**: ☐

Onto: ☐ **Not Onto**: ☐

(d) $\sim$ is the relation on $\mathbb{R}$ such that $x \sim y$ if and only if $xy = 1$

Symmetric: ☐ **Antisymmetric**: ☐

Reflexive: ☐ **Irreflexive**: ☐

Problem 6: Relations (8 points)

(a) Suppose that $\preceq$ is a partial order relation on a set A and let x be some element of A. Define what it means for x to be a maximal element of A. Careful: maximal not maximum.

(b) Let R be the relation on the set $\{A, B, C, D\}$ containing the following ordered pairs

$$(A, A), (B, A), (A, B), (B, C)$$

Draw a directed graph showing the transitive closure of R.

A	C
B	D

Problem 7: Algorithm analysis (10 points)

The following algorithm takes as input an arbitrary list of n real numbers a_1,...,a_n and reorders the numbers. The command **swap**(a_i, a_k) is used to interchange the positions of a_i and a_k in the list. If $i = k$, **swap**(a_i, a_k) does nothing.

procedure Reorder(a_1,..., a_n)
for $i := 1$ to $n - 1$
    $min := i$
    for $j := i$ to n
        if $a_j < a_{min}$ **then** $min := j$
    swap(a_i, a_{min})

1. If the initial list is $10, 5, 2, 8, 3$ what does the list look like after the outer loop has completed three iterations (i.e. after finishing the work for $i = 3$)?

2. How many comparisons does the algorithm use to reorder a list of n numbers? Express your answer as a function of n using big-theta notation and show how you derived that function.

Problem 8: Writing a proof (10 points)

Recall the definition of congruence mod m

$a \equiv b \pmod{m}$ if and only if there is an integer k such that $a - b = km$.

Using this definition plus high-school algebra, show that

If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $(a + c) \equiv (b + d) \pmod{m}$.

Justify the steps in your proof. Use only this definition and not other facts about modular arithmetic you may remember from class.

Problem 9: Induction (10 points)

Suppose that the function g is defined (on integers ≥ 1) by:

$$g(1) = 1$$

$$g(n) = g(n-1) + 6n - 6$$

Use induction to prove that $g(n) = 3n^2 - 3n + 1$, for every integer $n \geq 1$.

Problem 10: Harder proof (10 points)

Consider the set of intervals on the real line $J = \{(a, b) \mid a, b \in \mathbb{R} \text{ and } a < b\}$. The "precedes" relation P is defined on the set J as follows:

$$(a, b)\ P\ (c, d) \text{ if and only if } b < c$$

Prove that P is transitive.