# The "AM-HM" Inequality

## Scratchwork

If you just want to see the proof, skip to the next section. But here's the steps for coming up with the proof.

First let's build the outline and fill in the parts that are basically always the same:

We proceed by induction on $n$.

Our base case is when $n = 1$, and we have $(\sum_{i=1}^{1} x_i)(\sum_{i=1}^{1} \frac{1}{x_i}) = x_1 * \frac{1}{x_1} = 1 = 1^2$ ✓

Now suppose as our Inductive Hypothesis that for each $n$ from 1 up to some $k$, if $x_1 \cdots x_n$ are positive reals then $(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} \frac{1}{x_i}) \geq n^2$.

It remains to show that $(\sum_{i=1}^{k+1} x_i)(\sum_{i=1}^{k+1} \frac{1}{x_i}) \geq (k+1)^2$

$$\begin{aligned}(\sum_{i=1}^{k+1} x_i)(\sum_{i=1}^{k+1} \frac{1}{x_i}) &\geq \cdots \\ &\geq (k+1)^2\end{aligned}$$

Thus $(\sum_{i=1}^{k+1} x_i)(\sum_{i=1}^{k+1} \frac{1}{x_i}) \geq (k+1)^2$, QED.

So the remaining challenge will be filling in that algebra. With summations, you usually want to pull one term out of the summation:

$$\begin{aligned}(\sum_{i=1}^{k+1} x_i)(\sum_{i=1}^{k+1} \frac{1}{x_i}) &= [(\sum_{i=1}^{k} x_i) + x_{k+1}][(\sum_{i=1}^{k} \frac{1}{x_i}) + \frac{1}{x_{k+1}}] \\ &\geq \cdots \\ &\geq (k+1)^2\end{aligned}$$

You should always be looking for a way to apply the IH, which in this case means it would be nice if we could get a $(\sum_{i=1}^{k} x_i)(\sum_{i=1}^{k} \frac{1}{x_i})$ term to appear. This guides us to the next steps:

$$\begin{aligned}(\sum_{i=1}^{k+1} x_i)(\sum_{i=1}^{k+1} \frac{1}{x_i}) &= [(\sum_{i=1}^{k} x_i) + x_{k+1}][(\sum_{i=1}^{k} \frac{1}{x_i}) + \frac{1}{x_{k+1}}] \\ &= (\sum_{i=1}^{k} x_i)(\sum_{i=1}^{k} \frac{1}{x_i}) + (\sum_{i=1}^{k} x_i)(\frac{1}{x_{k+1}}) + (x_{k+1})(\sum_{i=1}^{k} \frac{1}{x_i}) + 1 \\ &\geq k^2 + (\sum_{i=1}^{k} \frac{x_i}{x_{k+1}}) + (\sum_{i=1}^{k} \frac{x_{k+1}}{x_i}) + 1 && \text{[by the IH]} \\ &\geq \cdots \\ &\geq (k+1)^2\end{aligned}$$

Your goal should be to make the two sides of the chain look as similar as possible. A $k^2$ and a $+1$ have appeared on top, so let's get those to appear on bottom as well, and bracket the remaining term on top in a compact form for comparison:

$$
\begin{aligned}
(\sum_{i=1}^{k+1} x_i)(\sum_{i=1}^{k+1} \frac{1}{x_i}) &= [(\sum_{i=1}^{k} x_i) + x_{k+1}][(\sum_{i=1}^{k} \frac{1}{x_i}) + \frac{1}{x_{k+1}}] \\
&= (\sum_{i=1}^{k} x_i)(\sum_{i=1}^{k} \frac{1}{x_i}) + (\sum_{i=1}^{k} x_i)(\frac{1}{x_{k+1}}) + (x_{k+1})(\sum_{i=1}^{k} \frac{1}{x_i}) + 1 \\
&\geq k^2 + (\sum_{i=1}^{k} \frac{x_i}{x_{k+1}}) + (\sum_{i=1}^{k} \frac{x_{k+1}}{x_i}) + 1 && \text{[by the IH]} \\
&= k^2 + [\sum_{i=1}^{k} (\frac{x_i}{x_{k+1}} + \frac{x_{k+1}}{x_i})] + 1 \\
&\geq \cdots \\
&\geq k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}
$$

Now for probably the hardest step of the proof: we have a summation on top and a $2k$ on bottom and we're trying to make them look the same, so we observe that $2k$ is the same as a summation of $k$ 2s. (This is the kind of step that I would never come up with by working from one end alone; it only becomes findable at all when working from both ends with the explicit goal of making them look as similar as possible at all costs.)

$$
\begin{aligned}
(\sum_{i=1}^{k+1} x_i)(\sum_{i=1}^{k+1} \frac{1}{x_i}) &= [(\sum_{i=1}^{k} x_i) + x_{k+1}][(\sum_{i=1}^{k} \frac{1}{x_i}) + \frac{1}{x_{k+1}}] \\
&= (\sum_{i=1}^{k} x_i)(\sum_{i=1}^{k} \frac{1}{x_i}) + (\sum_{i=1}^{k} x_i)(\frac{1}{x_{k+1}}) + (x_{k+1})(\sum_{i=1}^{k} \frac{1}{x_i}) + 1 \\
&\geq k^2 + (\sum_{i=1}^{k} \frac{x_i}{x_{k+1}}) + (\sum_{i=1}^{k} \frac{x_{k+1}}{x_i}) + 1 && \text{[by the IH]} \\
&= k^2 + [\sum_{i=1}^{k} (\frac{x_i}{x_{k+1}} + \frac{x_{k+1}}{x_i})] + 1 \\
&\geq k^2 + \sum_{i=1}^{k} 2 + 1 \quad \text{[because each } (\frac{x_i}{x_{k+1}} + \frac{x_{k+1}}{x_i}) \text{ is at least 2??]} \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}
$$

So we see that if only we could show that each $(\frac{x_i}{x_{k+1}} + \frac{x_{k+1}}{x_i})$ is at least 2, we'd be done. So we do that, giving us this final proof:

## The final proof

We first prove the following lemma: for positive $a, b$, $\frac{a}{b} + \frac{b}{a} \geq 2$. Proof:

$$\begin{aligned}
(a-b)^2 &\geq 0 && \text{[because any real squared is non-negative]} \\
a^2 - 2ab + b^2 &\geq 0 \\
a^2 + b^2 &\geq 2ab \\
\frac{a^2+b^2}{ab} &\geq 2 && \text{[because } ab \text{ is positive]} \\
\frac{a}{b} + \frac{b}{a} &\geq 2 \quad \square
\end{aligned}$$

Now we prove the main claim, that $\forall n \geq 1$, if $x_1 \cdots x_n$ are positive reals then $(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} \frac{1}{x_i}) \geq n^2$. We proceed by induction on $n$.

Our base case is when $n = 1$, and we have $(\sum_{i=1}^{1} x_i)(\sum_{i=1}^{1} \frac{1}{x_i}) = x_1 * \frac{1}{x_1} = 1 = 1^2$ ✓

Now suppose as our Inductive Hypothesis that for each $n$ from 1 up to some $k$, if $x_1 \cdots x_n$ are positive reals then $(\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} \frac{1}{x_i}) \geq n^2$.

It remains to show that $(\sum_{i=1}^{k+1} x_i)(\sum_{i=1}^{k+1} \frac{1}{x_i}) \geq (k+1)^2$

$$\begin{aligned}
(\sum_{i=1}^{k+1} x_i)(\sum_{i=1}^{k+1} \frac{1}{x_i}) &= [(\sum_{i=1}^{k} x_i) + x_{k+1}][(\sum_{i=1}^{k} \frac{1}{x_i}) + \frac{1}{x_{k+1}}] \\
&= (\sum_{i=1}^{k} x_i)(\sum_{i=1}^{k} \frac{1}{x_i}) + (\sum_{i=1}^{k} x_i)(\frac{1}{x_{k+1}}) + (x_{k+1})(\sum_{i=1}^{k} \frac{1}{x_i}) + 1 \\
&\geq k^2 + (\sum_{i=1}^{k} \frac{x_i}{x_{k+1}}) + (\sum_{i=1}^{k} \frac{x_{k+1}}{x_i}) + 1 && \text{[by the IH]} \\
&= k^2 + [\sum_{i=1}^{k} (\frac{x_i}{x_{k+1}} + \frac{x_{k+1}}{x_i})] + 1 \\
&\geq k^2 + \sum_{i=1}^{k} 2 + 1 && \text{[by the lemma above]} \\
&= k^2 + 2k + 1 \\
&= (k+1)^2
\end{aligned}$$

Thus $(\sum_{i=1}^{k+1} x_i)(\sum_{i=1}^{k+1} \frac{1}{x_i}) \geq (k+1)^2$, QED.