Induction Tutorial Solutions

11.1 Simple examples

b) We proceed by induction on $n$.

Base: Let $n = 1$. Then $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$.

Induction: Suppose (as our Inductive Hypothesis) that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for each $n \in \mathbb{Z}^+$ less than some positive integer $r$. Then our goal is to show $\sum_{k=1}^{r} \frac{1}{k(k+1)} = \frac{r}{r+1}$.

$$\sum_{k=1}^{r} \frac{1}{k(k+1)} = \sum_{k=1}^{r-1} \frac{1}{k(k+1)} + \frac{1}{r(r+1)}$$

(pulling a term out of the summation)

$$= \frac{(r-1)}{(r-1)+1} + \frac{1}{r(r+1)}$$

(by the Inductive Hypothesis)

$$= \frac{(r-1)}{r} + \frac{1}{r(r+1)}$$

(this and remaining steps are just algebra)

$$= \frac{(r-1)(r+1) + 1}{r(r+1)}$$

$$= \frac{r^2}{r(r+1)}$$

$$= \frac{r}{r+1}$$

Thus (by transitivity of equality) $\sum_{k=1}^{r} \frac{1}{k(k+1)} = \frac{r}{r+1}$, QED.

c) The IH on top of the equals sign below is a shorthand for showing where you are applying the Inductive Hypothesis.

Proof by induction on $n$.

Base: Let $n = 0$. Then $(\sum_{i=0}^{0} i)^2 = 0 = \sum_{i=0}^{0} i^3$. ✓

Induction: Fix $k$ and suppose that $(\sum_{i=0}^{n} i)^2 = \sum_{i=0}^{n} i^3$ for $n = 0, 1, \ldots, k - 1$. Then we
get the following:

\[
\begin{align*}
\left( \sum_{i=0}^{k} i \right)^2 &= \left( \sum_{i=0}^{k-1} i + k \right)^2 \\
&= \left( \sum_{i=0}^{k-1} i \right)^2 + 2k\left( \sum_{i=0}^{k-1} i \right) + k^2 \\
&\overset{IH}{=} \sum_{i=0}^{k-1} i^2 + 2k\left( \sum_{i=0}^{k-1} i \right) + k^2 \\
&= \sum_{i=0}^{k-1} i^2 + 2k\frac{(k-1)k}{2} + k^2 \\
&= \sum_{i=0}^{k-1} i^3 + k^3 \\
&= \sum_{i=0}^{k} i^3
\end{align*}
\]

Induction complete.

11.2 Induction with congruences

Fix \( a, b \in \mathbb{Z} \) and \( p \in \mathbb{Z}^+ \). Now we need to show \( \forall n \in \mathbb{Z}^+, P(n) \), where \( P(n) \) is “if \( a \equiv b \pmod{p} \) then \( a^n \equiv b^n \pmod{p} \)”. We proceed by induction on \( n \):

Base: We need to show \( P(1) \), i.e. that if \( a \equiv b \pmod{p} \) then \( a^1 \equiv b^1 \pmod{p} \). But this is clearly true since \( a = a^1 \) and \( b = b^1 \).

Induction: Fix \( z \), and suppose (as our Inductive Hypothesis) that for any \( i \) with \( 1 \leq i < z \), \( P(i) \) is true. Now we need to show \( P(z) \) is true, i.e. we need to show that if \( a \equiv b \pmod{p} \) then \( a^z \equiv b^z \pmod{p} \).

So suppose (towards direct proof) that \( a \equiv b \pmod{p} \). Using this fact along with \( P(z-1) \) (which is true by the IH), we also know that \( a^{z-1} \equiv b^{z-1} \pmod{p} \). Multiplying our two equivalences together gives us \( a \cdot a^{z-1} \equiv b \cdot b^{z-1} \pmod{p} \). This in turn gives us \( a^z \equiv b^z \pmod{p} \), QED.

(Commentary: notice that the original claim has four variables in it - \( a, b, n, p \). It would be valid to attempt an induction proof using any of those four as the induction variable, but if you pick something other than \( n \) in this case you will discover that there is no good way to finish the proof. So as always, don’t be afraid to switch tactics if your current path seems not to be working.)

11.4 A broken induction proof

(Commentary: Obviously the proof must be wrong since the claim it is proving is clearly false. While that is not enough to say where the flaw in the proof is, it does give us a good
place to check: $P(1)$ is true but $P(2)$ is false, so we should look at the inductive step and carefully audit its argument that $P(1) \rightarrow P(2).$

The argument implicitly relies on the fact that $S'$ and $S''$ are not disjoint. If the sets overlap by even one horse $H_*$, then the proof is correct that all horses in the union are the same color, since all the horses in $S'$ are $H_*$'s color and so are the horses in $S''$. However, consider the argument in the inductive step when $k = 2$. In this case, $S' = \{H_2\}$ and $S'' = \{H_1\}$, which are disjoint. Thus while it is true that all the horses in $S'$ are the same color and all the horses in $S''$ are the same color, it is wrong for the proof to claim from this that all the horses in the union must also be the same color.

MCS 5.16acdefh, aka Induction-Like Implications

Since a lot of people found the MCS textbook problem confusing, I wrote an equivalent problem which hopefully makes things clearer, and posted it in the tutorial problem list (https://courses.grainger.illinois.edu/cs173/fa2021/ALL-lectures/Tutorials/index.html). So if you were too confused by this problem, perhaps try that version before looking at the solutions here.

Since we’re going to need every single $P$ we come up with to satisfy $\forall k, P(k) \rightarrow P(k+2)$, let’s first look at a few examples of such predicates. Remember that the only time $P(k) \rightarrow P(k+2)$ is false is if $P(k)$ is true AND $P(k+2)$ is false. We thus have a lot of leeway when designing $P$, including useful simple extremes where $P$ is always false or always true. Also, for any $z$, $P(n) = n \geq z$ will work, because then whenever $P(k)$ is true, $k+2$ is bigger than $k$ so $P(k+2)$ is also true. Other predicates that work include “$n$ is even”, “$n$ is odd”, and a conjunction or disjunction (“and” and “or”) of any two other predicates that work.

Now let’s look at the individual statements:

(Translation between the two versions of this problem: we say that a statement “always holds” if it is true for every (valid) $P$, i.e. every answer to my prompt 1 is correct (as long as it has the $\forall k, P(k) \rightarrow P(k+2)$ property), and there is no $P$ we can choose for prompt 2 to make it false. Similarly, we say a statement “never holds” if every answer to my prompt 2 is correct and there is no $P$ we can choose for prompt 1 to make it true, and it “can hold” if there’s a $P$ for both prompts. Part (a) in MCS became the example in my version.)

a) Can hold - true if $P(n)$ is simply the predicate “True” (i.e. it’s true regardless of the value of $n$), false for $P(n) := “False”$ (or $P(n) := “n$ is even”; I’ll just give one answer from now on but there are usually many other options). (read “:=” as “is defined to be”)

c) Can hold - true for $P(n) := “False”$, false for $P(n) := “True”$.

d) Never holds - from $(\forall n \leq 100(P(n))$ we know $P(100)$ would have to be true, but from $(\forall n > 100(\neg P(n)))$ we know $P(102)$ would have to be false. Thus there is no way to make this statement true while sticking to the rule that $P(k) \rightarrow P(k+2)$.

e) Can hold - true for $P(n) := “n \geq 101”$, false for $P(n) := “False”$.

f) Can hold - true for $P(n) := “True”$, false for $P(n) := “n$ is even”. Note that $\forall n(P(n+2))$ is equivalent (in our given universe of nonnegative integers) to $\forall n \geq 2(P(n))$.
h) Always holds - regardless of what \( P \) is chosen (as long as it follows the \( P(k) \rightarrow P(k + 2) \) rule), we have one of two cases:

- \( P(1) \) is false. In this case the whole statement is vacuously true.
- \( P(1) \) is true. In this case, \( P(3) \) must also be true (since \( P(1) \rightarrow P(1 + 2) \)), then \( P(5) \) must also be true (since \( P(3) \rightarrow P(3 + 2) \)), etc for all the odd natural numbers, so we get that \( P \) is true for all odd naturals, i.e. for any number that can be written as \( 2n + 1 \) for some \( n \in \mathbb{N} \). (Formalizing this argument would require induction, though it would be a bit awkward to formalize since our goal would be to induct over only the odd numbers.)

The Diagonal Robot

Let \( P(n) \) be the claim “After the robot takes \( n \) steps, it is guaranteed to be at a location \((x, y)\) with \( x + y \) even.” If we can prove that \( P(n) \) is true for all \( n \in \mathbb{N} \), then we are done, because \((0, 1)\) has an odd sum of coordinates. So we proceed by induction on \( n \):

Base: After the robot takes zero steps, it is still at its starting location of \((1, 1)\), and \( 1 + 1 \) is even.

Induction: Suppose \( P(n) \) is true for \( n = 0, 1, \cdots, k \). Then we need to show \( P(k + 1) \), i.e. that after the robot takes \( k + 1 \) steps, it is still guaranteed to be at a location with even coordinate sum. So consider an arbitrary case where the robot has taken \( k + 1 \) steps. One step earlier, it had taken \( k \) steps, and by the inductive hypothesis, it was at a location \((x, y)\) with \( x + y \) even. So now after the \((k + 1)\)st step, the only four places it can be are \((x + 1, y + 1)\), \((x + 1, y - 1)\), \((x - 1, y + 1)\), and \((x - 1, y - 1)\). In each of those cases the sum is either \( x + y + 2 \), \( x + y \), or \( x + y - 2 \), so since \( x + y \) is even, the new sum is also even.