Countability Tutorial Solutions

19.1 Which Kind of Infinity?

A common fast way to show that a set is countable is to note that every element in the set has a finite representation. Also you may use the fact that there are no infinite sets with smaller cardinality than \( \mathbb{N} \), so if you can show some set \( X \) is infinite and \( |X| \leq |\mathbb{N}| \), then \( |X| = |\mathbb{N}| \).

a) **Countably infinite.** In fact it’s basically the definition of countably infinite - the bijection mapping it to \( \mathbb{N} \) is \( \text{id}_{\mathbb{N}} \).

b) **Uncountable.** The powerset of a set always has a (strictly) larger cardinality than that set. (Or a handwavy ‘solution’ thinking about representations: these do not appear to all have finite representations - if I have an infinite set of naturals with no pattern, how would I possibly write down that set?)

c) **Uncountable.** We know \( \mathbb{R} \) is uncountable, and \( \mathbb{R} \subseteq \mathbb{C} \).

d) **Countably infinite.** We can provide a one-to-one function \( f \) mapping these to the (finite) bit strings: given \( S \) with maximum element \( n \), return the bit string of length \( n + 1 \) with a 1 in (0-indexed) position \( i \) iff \( i \in S \). For example, \( f(\{0, 3, 4\}) = 10011 \). And we know the set bit strings (or any other strings with a finite alphabet) are countable. (Thinking with representations: each \( S \in X \) has a roster notation which is finite - e.g. \( \{0, 3, 4\} \).)

e) **Countably infinite.** Each book is just one finite string using a finite alphabet. (You may be tempted to think of a book as a list of strings separated by spaces, but that’s making it more complicated than necessary - there’s no need to treat characters like space and newline any differently from a and b.)

f) **Countably infinite.** We know \( \mathbb{Q} \) is countable, and this set is a subset of \( \mathbb{Q} \). (Thinking with representations: these are reals specifically chosen to have expansions that end - i.e. representations that are finite.)

19.2 A Curious Bijection

a)
b) Consider the values of $x, y$ satisfying $x + y = k$.

Because we are in $\mathbb{N}$, for any such values of $x$ and $y$ we have that $y \geq 0$ and therefore $x \leq k$. For any value $x \leq k$, we can let $y = k - x$ to achieve $x + y = k$.

Thus, $x$ ranges from 0 to $k$, and $f(x, y) = s(x + y) + x = s(k) + x$ ranges from $s(k)$ to $s(k) + k$. Remembering from lecture that $s(k) = \frac{k(k+1)}{2}$, we can also write this as:

$$\frac{k(k+1)}{2} \leq f(x, y) \leq \frac{k(k+1)}{2} + k$$

c) The preimage of 17 is $\{(2, 3)\}$. Note that $f(2, 3) = s(5) + 2 = 15 + 2 = 17$.

We can show that $(2, 3)$ is the only element in the pre-image by noting from our solution to part d) that, for all $x, y$, if $f(x, y) = f(2, 3)$, then $x + y = 2 + 3 = 5$. Testing all such values of $x$ and $y$ shows that $(2, 3)$ is the only element in the pre-image of 17.

(Alternatively, we could argue that there can’t be any other element in the pre-image because, as demonstrated through parts (d) and (e), $f$ is one-to-one.)

d) Let $k = x + y$, $l = p + q$. From the given supposition we know $k \neq l$, so without loss of generality, assume that $k < l$.

We get the following:

$$f(x, y) \leq \frac{k(k+1)}{2} + k$$

[from part (b)]

$$= \frac{k^2 + 3k}{2}$$

$$< \frac{k^2 + 3k + 2}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$$\leq \frac{l(l+1)}{2}$$

[k < l, and $k, l \in \mathbb{Z}$, so $k + 1 \leq l$]

$$\leq f(p, q)$$

[from part (b)]
This establishes $f(x, y) < f(p, q)$, so $f(x, y) \neq f(p, q)$; QED.

e) Suppose not. That is, suppose towards a proof by contradiction that $f(x, y) = f(p, q)$. Further, let $k = x + y = p + q$. Then:

\[
\begin{align*}
  f(x, y) &= f(p, q) \\
  s(x + y) + x &= s(p + q) + p \\
  s(k) + x &= s(k) + p \\
  x &= p
\end{align*}
\]

Since $x = p$ and $x + y = p + q$, we have that $y = q$. But we assumed that $(x, y) \neq (p, q)$, contradiction. So our initial supposition must be false, and thus instead we know $f(x, y) = f(p, q)$; QED.

**Additional problem**

Lemma: For (non-empty) sets $A$ and $B$, there exists a one-to-one function $f : A \to B$ if and only if there exists an onto function $g : B \to A$.

Proof: See solution to the “additional tutorial problem” from the Functions week - the only difference is that now we are working with arbitrary sets instead of subsets of $\mathbb{N}$, so where that solution uses the function $\text{minimum}$ (which can choose a representative from a set of naturals), we instead have to use the choice function $h$ from the hint. □

We know that by definition, there exists a one-to-one function $f : A \to B$ if and only if $|A| \leq |B|$. So now by the lemma, we’ve established that there exists an onto function $g : B \to A$ if and only if $|A| \leq |B|$. 
