Worksheet on Sets, Functions, and Relations

Benjamin Cosman, Patrick Lin and Mahesh Viswanathan

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Definitions from the Lecture

- $\emptyset$ is the empty set, $\mathbb{Z}$ is the set of integers, $\mathbb{N}$ is the set of natural numbers (includes 0), and $\mathbb{R}$ is the set of real numbers.
- $R \subseteq S$ holds if $\forall x\ (x \in R \to x \in S)$ and $R = S$ if $R \subseteq S$ and $S \subseteq R$.
- Set operations are defined as follows. We assume that U is the universe (i.e., $R \subseteq U$ and $S \subseteq U$).

$$R \cup S = \{x \mid x \in R \ \vee \ x \in S\} \qquad R \cap S = \{x \mid x \in R \ \wedge \ x \in S\}$$
$$R \setminus S = \{x \in R \mid x \notin S\} \qquad \overline{R} = U \setminus R$$
$$R \times S = \{(r,s) \mid r \in R \ \wedge \ s \in S\} \qquad \mathcal{P}(R) = \{A \mid A \subseteq R\}$$

- For a function $f : A \to B$, A is the domain ($\mathsf{dom}(f)$), B is the codomain ($\mathsf{codom}(f)$), and $\{f(x) \mid x \in A\}$ is the range ($\mathsf{rng}(f)$).
- Function $f : A \to B$ is surjective/onto if $\mathsf{rng}(f) = B$ or $\forall y \in B \exists x \in A\ (f(x) = y)$.
- Function $f : A \to B$ is injective/1-to-1 if $\forall x, y \in A\ (x \neq y \to f(x) \neq f(y))$ or (its contrapositive) $\forall x, y \in A\ (f(x) = f(y) \to x = y)$.
- A binary relation R with domain A and codomain B is a subset of $A \times B$.

Problem 1. Let consider the following sets.

$$A = \{0,2,4,6\} \quad B = \{\{0\},\{2\},\{4\},\{6\}\} \quad C = A \cup \emptyset$$
$$D = A \cup \{\emptyset\} \quad E = \{n \in \mathbb{N} \mid n^2 \in \mathbb{N}\} \quad F = \{n^2 \in \mathbb{N} \mid n \in \mathbb{N}\}$$

Answer the following questions about these sets.

1. What are the elements of sets A, B, C, and D?
2. Which of the following are true? $\emptyset \in A$, $\emptyset \in B$, $\emptyset \in C$, $\emptyset \in D$.
3. Which of the following are true? $0 \in A$, $0 \in B$, $\{0\} \in A$, $\{0\} \in B$.
4. Which of the following are true? $0 \in E$, $2 \in E$, $\{0\} \in E$, $\{2\} \in E$, $0 \in F$, $2 \in F$, $\{0\} \in F$, $\{2\} \in F$

5. Is $\{\} = \{\emptyset\}$?
6. Which of the following are true? $\emptyset \subseteq A$, $\emptyset \subseteq B$, $\{0\} \subseteq A$, $\{0\} \subseteq B$.
7. Which of the following are true? $A \subseteq E$, $B \subseteq E$, $A \subseteq F$, $B \subseteq F$, $E \subseteq F$, $F \subseteq E$.
8. What are the sets $B \cup C$ and $B \cup D$?
9. What is the set $A \cap B$?
10. What are the sets $B \setminus A$, $C \setminus A$ and $D \setminus A$?
11. Use the set builder notation to describe the set $E \setminus F$.
12. What are the sets $\emptyset \times B$, $\emptyset \times D$, and $\emptyset \times E$?
13. What are the sets $A \times B$ and $B \times A$? Are these two sets equal?
14. What are the sets $\mathcal{P}(\emptyset)$ and $\mathcal{P}(\{\emptyset\})$?
15. Which of the following are true? $\emptyset \in \mathcal{P}(A)$, $\emptyset \subseteq \mathcal{P}(A)$, $\{0\} \in \mathcal{P}(A)$, $\{0\} \subseteq \mathcal{P}(A)$.

Problem 2. For any sets A, B, C, prove that $(A \setminus C) \setminus (B \setminus C) \subseteq (A \setminus B)$.

Problem 3. Let $f : A \to B$ and $g : B \to C$. Suppose $g \circ f$ is injective.

1. Prove that f is also injective.
2. Is g necessarily injective? Justify your answer.