Lecture Summary

- The pigeon hole principle says that if $|A| > |B|$ then for any function $f : A \rightarrow B$ there are $a, b \in A$ such that $f(a) = f(b)$.

- The generalized pigeon hole principle is as follows. Let $A$ be a set and $B$ be an $n$-element set (say) $\{b_1, b_2, \ldots, b_n\}$. Let $q_1, \ldots, q_n$ be $n$ natural numbers such that

$$|A| > q_1 + q_2 + \cdots + q_n.$$ 

For any function $f : A \rightarrow B$ there is an $i \in \{1, 2, \ldots, n\}$ such that $|\{a \in A \mid f(a) = b_i\}| > q_i$.

- Observe that the (basic) pigeon hole principle is a special case of the generalized pigeon hole principle, where each $q_i = 1$.

- Another special case of the generalized pigeon hole principle is as follows. If $|A| > k|B|$ then for any function $f : A \rightarrow B$ there are $k + 1$ elements $a_1, a_2, \ldots, a_{k+1} \in A$ such that $f(a_i) = f(a_j)$ for any $i, j \in \{1, 2, \ldots, k+1\}$.

- The principle of inclusion-exclusion says that for any sets $S_1, S_2, \ldots, S_n$,

$$\left| \bigcup_{i=1}^{n} S_i \right| = \sum_{\varnothing \neq I \subseteq \{1, 2, \ldots, n\}} (-1)^{|I|+1} \left| \bigcap_{i \in I} S_i \right|$$

- When $n = 2$ or $n = 3$, the principle of inclusion-exclusion specializes to the following equations.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

**Problem 1.** Prove that any subset $A \subseteq \{1, 2, \ldots, 9\}$ of size 6, must contain a pair of numbers whose sum is 10.

**Problem 2.** In a group of $n$ people, prove that there are two people with the same number of friends.
Problem 3. For any sequence of integers \( a_1, a_2, \ldots, a_n \), prove that there is some “consecutive sum” that is divisible by \( n \). That is, prove that there are indices \( 0 \leq i < j \leq n \) such that \( n | (a_{i+1} + a_{i+2} + \cdots + a_j) \).

Problem 4. Let \( S \) be the set of permutations of \( \{0, 1, 2, \ldots, 9\} \) such that either 7 and 3, or 1 and 7 appear consecutively (in that order). For example, \( 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \not\in S \) (as neither 1.7 nor 7, 3 appear), \( 0, 2, 3, 4, 5, 6, 8, 9, 7, 1 \not\in S \) (the 7, 1 at the end does not count as they are in the flipped order), \( 0, 2, 3, 4, 5, 6, 8, 9, 1, 7 \in S \) (because of the 1, 7 at the end), \( 1, 7, 3, 0, 2, 4, 5, 6, 8, 9 \in S \) (because of either 1 followed by 7 or 7 followed by 3). What is \( |S| \)?

Problem 5. How many ways can we place 4 distinct letters in 4 different pre-addressed envelopes so that no letter is placed the correct envelope?