Homework on Directed Graphs

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Problem 1. A binary relation R on a set A is *asymmetric* if $\forall x, y \in A(xRy \rightarrow \neg yRx)$.

a) Prove that for any digraph G, if $E(G)$ is asymmetric and transitive then G is acyclic.[1]

b) Prove that the converse is not true.[2]

Problem 2. Determine whether each relation below is transitive. Prove your answers.

a) $R = \{(a, b) | a, b \in \mathbb{N} \wedge a < 2b\}$

b) $R = \{(a, b) | a, b \in \mathbb{N} \wedge a < b/2\}$

Problem 3. Describe an algorithm for picking vertices from a DAG one at a time such that no vertex has an edge leading to a vertex that was chosen earlier in the process.[3] Prove that your algorithm works.

Problem 4. Consider a digraph representing course prerequisites: each vertex is a course, and there is an edge from course A to course B if A is a prerequisite for B. (Assume for simplicity that there are no "or" requirements - i.e. CS 225 can require CS 173 and CS 125 but it can not require "CS 173 or Math 213"). A list of classes $(c_1, c_2, ..., c_n)$ is *allowable* if, when you take those classes in that order, you are never missing the prerequisites for any class you take. Prove that if there is a cycle in the graph, then there is no allowable list of all the classes.

Problem 5. Prove the following theorem:

Theorem 1. *A directed graph has a topological sort if and only if it is acyclic.*[4]

[1] Hint: You may wish to prove this lemma first: for any digraph H, if $E(H)$ is transitive and H has any cycles then H has a self-loop (i.e. a (v, v) edge).

[2] Recall: the converse of $p \rightarrow q$ is $q \rightarrow p$

[3] Hint: Use Problem 6 from the worksheet. How would you choose your classes so that you never skip a prereq?

[4] Hint: You've already done all the hard work in earlier worksheet and homework problems