Homework on Cardinality

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Problem 1. For each of the following sets, determine if it is finite, infinite but countable, or uncountable. There is no need to prove your result.

(a) $\mathcal{P}(\mathbb{Q})$ where $\mathbb{Q}$ is the set of rational numbers

(b) $\{x \in \mathbb{R} \mid x^2 \leq 0\}$

(c) $\{3n \mid n \in \mathbb{N}\}$

(d) $\{n \in \mathbb{N} \mid n^2 = n^3\}$

(e) $\mathbb{R}$

(f) $A \cap B$, where A and B are countable sets.

Problem 2. Recall that $\mathbb{E} = \{2n \mid n \in \mathbb{N}\}$ is the set of even natural numbers. Let $\mathsf{Sq} = \{n^2 \mid n \in \mathbb{N}\}$ be the set of perfect square natural numbers. Show that $|\mathbb{E}| = |\mathsf{Sq}|$ by describing a bijective function between the sets and proving your function to be bijective.

Problem 3. Prove that $D = \{(m, n) \in \mathbb{N} \times \mathbb{N} \mid m \leq n\}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality.

Problem 4. Recall that the interval $(0, 1) = \{r \in \mathbb{R} \mid 0 < r < 1\}$ and $[0, 1) = \{r \in \mathbb{R} \mid 0 \leq r < 1\}$. Prove that $|(0, 1)| = |[0, 1)|$. *Hint:* It is difficult to find a bijective function between these sets. Use the Cantor-Schröder-Bernstein Theorem, Proposition 5 from notes, and Proposition 11 from notes/Problem 3 in worksheet.

Problem 5. Suppose A, B are infinite sets that are countable. Prove that

(a) $A \times B$ is countable. *Hint:* Consider showing $|A \times B| = |\mathbb{N} \times \mathbb{N}|$.

(b) If A and B are disjoint (i.e., $A \cap B = \emptyset$) then $A \cup B$ is countable. *Hint:* Consider adapting the proof that shows $|\mathbb{Z}| = |\mathbb{N}|$.

Based on part (b) above and the fact that $\mathbb{R}$ are uncountable [1], what can you conclude about the cardinality of the irrational numbers. Is it countable or uncountable?

[1] This was not proved in class. One needs to use diagonalization to prove this.

Problem 6. Recall that $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$. Also $(0, \infty) = \{x \in \mathbb{R} \mid 0 < x\}$.

(a) Prove that $h : (0, \infty) \to (0, 1)$ defined as $h(x) = \frac{x}{x+1}$ is a bijection. Thus $|(0, 1)| = |(0, \infty)|$.

(b) Prove that $|\mathbb{R}| = |(0,1)|$. *Hint:* Can you adapt the bijection in part (a) to map non-negative numbers to $[\frac{1}{2}, 1)$ and negative numbers to $(0, \frac{1}{2})$?