Homework on Sets, Functions and Relations

Benjamin Cosman, Patrick Lin and Mahesh Viswanathan

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Problem 1. Let A, B, and C be arbitrary sets. Prove that if $A \subseteq B$ then $C \setminus B \subseteq C \setminus A$.

Problem 2. Prove that for any sets A, B, C, $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

Problem 3. Let $f : A \to B$ be an arbitrary function. For a subset $S \subseteq A$, define $f(S) = \{f(a) \mid a \in S\}$. Prove that for any $S, T \in \mathcal{P}(A)$, if $S \subseteq T$ then $f(S) \subseteq f(T)$.

Problem 4. Consider functions $f : A \to B$ and $g : B \to C$.

(a) Prove that if f and g are surjective then $g \circ f$ is also surjective.

(b) Prove that if the composition $g \circ f$ is bijective then f is injective and g is surjective.

(c) Give an example of functions f and g such that f is bijective but $g \circ f$ is not bijective.

Problem 5. Suppose $f : \mathbb{N} \to \mathbb{N}$ is surjective (onto). Define $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as

$$g(x, y) = f(x)f(y).$$

Prove that g is surjective (onto).

Problem 6. Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is injective (1-to-1). Define $g : \mathbb{Z} \to (\mathbb{Z} \times \mathbb{Z})$ such that

$$g(x) = (2f(x), |f(x)|).$$

Prove that g is injective (1-to-1).

Problem 7. The language of sets and relations have a close connection to relational databases, implemented by software packages as MySQL. In this problem you are asked to manipulate and analyze a large data set using operations over sets and relations.

Consider a basic Web search engine, which stores information on Web pages and processes queries to find pages satisfying conditions provided by users. The following sets and relations encapsulate this information at a high level.

- P is the set of *pages* the search engine knows about.
- L (*links*) is a binary relation over pages, where $(p_1, p_2) \in L$ iff page p_1 has a link to p_2.

- E is the set of *endorsers* who have recorded their opinions about which pages are high quality.
- R (*recommends*) is a binary relation between endorsers and pages such that $(e, p) \in R$ iff endorser e recommends page p.
- W is the set of *words* that appear on pages.
- M (*mentions*) is a binary relation between pages and words such that $(p, w) \in M$ iff page p mentions word w.

Operations on sets and relations we will consider are the following.

- set union $\cup$, set intersection $\cap$, and set difference $\setminus$
- For a relation $Q \subseteq A \times B$ and subset $S \subseteq A$, $Q(S)$ is the *relational image*, defined as $\{b \mid \exists a \in S\ (a, b) \in Q\}$
- For a relation $Q \subseteq A \times B$, its *relational inverse* is $Q^{-1} = \{(b, a) \mid (a, b) \in Q\} \subseteq B \times A$
- Finally, for relations $P \subseteq B \times C$ and $Q \subseteq A \times B$, their *relational composition* $P \circ Q \subseteq A \times C$ is the relation $\{(a, c) \mid \exists b \in B\ ((a, b) \in Q \ \wedge\ (b, c) \in P)\}$

For each of the queries below, describe the result of the query using the sets, relations and operations defined above. For example, the query "The set of pages containing the word logic" can be described by the expression $M^{-1}(\{\mathsf{logic}\})$,

(a) The set of pages containing the word logic but not the word predicate.

(b) The set of pages containing the word set recommended by Cosman.

(c) The relation that relates endorser e and word w iff e has recommended a page containing w.

(d) The set of endorsers who have recommended pages containing the word algebra.

(e) The relation that relates word w to page p iff w appears on a page that has a link to p.