Homework on Summations and Recurrences

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Fall 2020

Problem 1 (Infinite geometric series)**.** For an infinite sequence $a_0, a_1, \ldots$, define $\sum_{i=0}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=0}^{n} a_i$. [1]

Prove that for $|r| < 1$, $\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$.

[1] Yes, that is a limit. Calculus is a prerequisite for this course :)

Problem 2 (Arithmetico-Geometric Series)**.**

a) Compute a closed form for $\sum_{i=1}^{n} ix^i$. [2]

b) Prove that if $|x| < 1$, $\sum_{i=0}^{\infty} ix^i = \frac{x}{(1-x)^2}$. [3]

[2] Hint: Set $S = \sum_{i=1}^{n} ix^i$, then compute $(1-x)S$.

[3] See Problem 1 for the definition of an infinite sum.

Problem 3 (Rolling down the river)**.** Compute a closed form for the following recurrence, defined over non-negative numbers.

$$J(n) = \begin{cases} 1 & \text{if } n \in \{0,1\} \\ \frac{1}{2}J(n-2)+1 & \text{if } n > 1 \end{cases}.$$

Problem 4 (Reoccuring recurrences)**.** Using the recursion tree method, compute closed forms for the following recurrences, both of which are defined over non-negative powers of two (i.e., 2^i for $i > 0$).

i) $B(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2B(\frac{n}{2}) + n & \text{otherwise} \end{cases}$

ii) $C(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2C(\frac{n}{2}) + n^2 & \text{otherwise} \end{cases}$