Lecture 8: Functions, Binary Relations, and Cardinality

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Functions: A function \( f : A \to B \) assigns to an element of one set the domain (in this case \( A \)), an element from another set the codomain (in this case \( B \)).

Example 1. \( \text{inc} : \{0, 1, 2\} \to \{0, 1, 2\} \) where \( \text{inc}(0) = 1 \), \( \text{inc}(1) = 2 \), and \( \text{inc}(2) = 0 \).

\( \text{dbl} : \mathbb{N} \to \mathbb{N} \) where \( \text{dbl}(n) = 2n \).

\( \text{twoinc} : \mathbb{Z} \to \mathbb{Z} \) where \( \text{twoinc}(x) = x + 2 \).

\( \text{sq} : \mathbb{R} \to \mathbb{R} \) where \( \text{sq}(x) = x^2 \).

Evaluation on Sets: Given a function \( f : A \to B \) and \( S \subseteq A \), \( f(S) = \{ f(n) \mid n \in S \} \subseteq B \).

Example 2. \( \text{inc}(\{0, 1\}) = \{1, 2\} \)

\( \text{dbl}([n]) = \{2 \mid n \in \mathbb{N} \} = \{2n \mid n \in \mathbb{N} \} = 2\mathbb{N} \)

Range: The range of \( f : A \to B \) is the set \( f(A) \).

\[ \text{range}(f) \subseteq B \]

Surjective/Onto: \( f : A \to B \) is surjective/onto if \( \text{range}(f) = f(A) = B = \text{codomain}(f) \), i.e.,

\[ \forall y \in B \exists x \in A(f(x) = y) \]

Question 1. Which of the following functions is surjective?
(a) \( \text{inc} \)
(b) \( \text{twoinc} \)
(c) \( \text{sq} \)
(d) \( \text{dbl} \)

\( \text{inc}(\{0, 2\}) = \{1\} \)

\( \text{twoinc}([n]) = \{2 \mid n \in \mathbb{Z} \} = \{2n \mid n \in \mathbb{Z} \} = 2\mathbb{Z} \)

\( \text{sq}(\{0, 2\}) = \{0, 4\} \)

\( \text{dbl}([n]) = \{2n \mid n \in \mathbb{N} \} = 2\mathbb{N} \)

Injective/1-to-1: \( f : A \to B \) is injective/1-to-1 if distinct elements get mapped to distinct elements, i.e.,

\[ \forall x \in A \forall y \in A(\langle x \neq y \rangle \text{ IMPLIES } (f(x) \neq f(y))) \]

\( \Rightarrow \) CONTRAPosition: \( \forall x \in A \ \forall y \in A(\langle f(x) = f(y) \rangle \text{ IMPLIES } x = y) \)

Question 2. Which of the following functions is injective?
(a) \( \text{inc} \)
(b) \( \text{twoinc} \)
(c) \( \text{sq} \)
(d) \( \text{dbl} \)

let \( x, y \) be arbitrary elements s.t. \( \text{dbl}(x) = \text{dbl}(y) \Rightarrow 2x = 2y \Rightarrow x = y \)

Composition: For functions \( f : A \to B \) and \( g : B \to C \), the composition \( g \circ f \) is the function \( A \to C \) defined as \( (g \circ f)(x) = g(f(x)) \), for all \( x \in A \).

Problem 1. If \( f : A \to B \) and \( g : B \to C \) are injective then \( g \circ f \) is injective.

(consider \( x, y \) \( \in A \). Assume \( g(f(x)) = g(f(y)) = g(y) \)
\( \Rightarrow f(x) = f(y) \)
\( \Rightarrow x = y \)

\( \Rightarrow \) \( f \) and \( g \) injective)

Proposition 1. If \( f : A \to B \), \( g : B \to C \), and \( g \) is surjective then \( g \circ f \) is surjective.

Bijective: A function that is injective/1-to-1 and surjective/onto.
\[ R \subseteq A_1 \times A_2 \times A_3 \times \cdots \times A_n = \{ \langle a_1, a_2, \ldots, a_n \rangle \mid \forall i, a_i \in A_i \} \]

Binary Relation: \( R \subseteq A \times B \), where \( A \) is the domain, and \( B \) is the codomain.

\[ = \{ \langle a, b \rangle \mid a \in A \text{ and } b \in B \} \]

Notation: \((a, b) \in R\) or \(aRb\) or \(R(a, b)\)

Example 3. For any function \( f : A \to B \), \( \text{graph}(f) = \{(x, f(x)) \mid x \in A\} \).

“less than” is a binary relation from \( \mathbb{R} \) to \( \mathbb{R} \).

Consider the relation teaches \( \subseteq \) Instructor \( \times \) Courses. It may have tuples of the form

\((\text{viswanathan, CS173L1}), (\text{viswanathan, CS173L1}), (\text{forbes, CS473}), (\text{chekuri, CS473}) \ldots\)

Cardinality (of finite sets): \(|X| = \) number of elements in \( X \).

Example 4. \(|\emptyset| = \) \(|\{0,1,2,3\}| = \) \(|\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}| = \) \(|\{0,1,1,2,2\}| = \)

\(|\{0,1,2\} \times \{a,b,c\}| = \)

Proposition 2. The following statements hold for finite sets \( A \) and \( B \).

1. If there is a surjective function \( f : A \to B \) then \(|A| \geq |B|\).
2. If there is a injective function \( f : A \to B \) then \(|A| \leq |B|\).
3. If there is a bijective function \( f : A \to B \) then \(|A| = |B|\).

Proposition 3. For a set \( A \) such that \(|A| = n\), \(|\text{pow}(A)| = 2^n\).