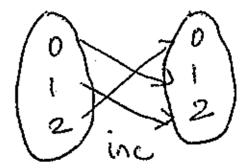
Lecture 8: Functions, Binary Relations, and Cardinality

Date: September 13, 2019.

Functions: A function $f : A \to B$ assigns to an element of one set the **domain** (in this case A), an element from another set the **codomain** (in this case B). | partial fn $f : A \hookrightarrow B$

Example 1. $\mathsf{inc} : \{0,1,2\} \to \{0,1,2\}$ where $\mathsf{inc}(0) = 1$, $\mathsf{inc}(1) = 2$, and $\mathsf{inc}(2) = 0$.



$\mathsf{dbl} : \mathbb{N} \to \mathbb{N}$ where $\mathsf{dbl}(n) = 2n$.

$\mathsf{twoinc} : \mathbb{Z} \to \mathbb{Z}$ where $\mathsf{twoinc}(x) = x + 2$. | $\mathsf{ntwoinc} : \mathbb{N} \to \mathbb{N}$.

$\mathsf{sq} : \mathbb{R} \to \mathbb{R}$ where $\mathsf{sq}(x) = x^2$. $\underbrace{\mathsf{ntwoinc}(n) = n+2}_{\text{not onto}}$

Evaluation on Sets: Given a function $f : A \to B$ and $S \subseteq A$, $f(S) = \{f(n) \mid n \in S\} \subseteq B$.

Example 2. $\mathsf{inc}(\{0,1\}) = \{1, 2\}$
$\mathsf{dbl}(\mathbb{N}) = \{ n \mid n \text{ is even and } n \in \mathbb{N} \} = \{ 2n \mid n \in \mathbb{N} \} = 2\mathbb{N}$

| $\mathsf{twodec}(n) = n-2$
| $\mathsf{twodec} : \mathbb{N} \to \mathbb{N}$
| $\mathsf{inv} : \mathbb{R} \hookrightarrow \mathbb{R}$
| $\mathsf{inv}(x) = \frac{1}{x}$

Range: The range of $f : A \to B$ is the set $f(A)$. $\mathsf{range}(f) \subseteq B$

Surjective/Onto: $f : A \to B$ is surjective/onto if $\mathsf{range}(f) = f(A) = B = \mathsf{codomain}(f)$, i.e.,

$$\forall y \in B \, \exists x \in A (f(x) = y)$$

Question 1. Which of the following functions is surjective? (a) inc, (b) dbl, (c) twoinc, (d) sq

$\mathsf{inc}(\{0,1,2\}) = \{1,2,0\} = \mathsf{codomain}(\mathsf{inc})$ $\mathsf{range}(\mathsf{sq}) = \{ x \in \mathbb{R} \mid x \geq 0 \}$

Injective/1-to-1: $f : A \to B$ is injective/1-to-1 if *distinct* elements get mapped to *distinct* elements, i.e.,

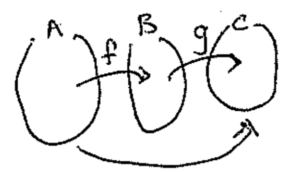
$$\forall x \in A \, \forall y \in A ((x \neq y) \text{ IMPLIES } (f(x) \neq f(y)))$$

$\to$ contrapositive : $\forall x \in A \ \forall y \in A.$ $f(x) = f(y)$ IMPLIES $x = y$

Question 2. Which of the following functions is injective? (a) inc, (b) dbl, (c) twoinc, (d) sq

Let x, y be arbitrary elements s.t $\mathsf{dbl}(x) = \mathsf{dbl}(y) \Rightarrow 2x = 2y \Rightarrow x = y$

Composition: For functions $f : A \to B$ and $g : B \to C$, the composition $g \circ f$ is the function $A \to C$ defined as $(g \circ f)(x) = g(f(x))$, for all $x \in A$.



Problem 1. If $f : A \to B$ and $g : B \to C$ are injective then $g \circ f$ is injective.

Consider $x, y \in A$. Assume $g(f(x)) = g \circ f(x) = g \circ f(y) = g(f(y))$
g is 1-to-1 : $\Rightarrow$ $f(x) = f(y)$
f is 1-to-1 : $\Rightarrow$ $x = y$

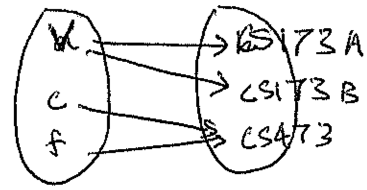
Proposition 1. *If $f : A \to B$, $g : B \to C$, and* f and *g is surjective then $g \circ f$ is surjective.*

Bijective: A function that is injective/1-to-1 **and** surjective/onto.

$R \subseteq A_1 \times A_2 \times A_3 \times \cdots A_n = \{(a_1, a_2 \ldots a_n) \mid \forall i \; a_i \in A_i\}$

Binary Relation: $R \subseteq A \times B$, where A is the *domain*, and B is the *codomain*.

$= \{(a,b) \mid a \in A \text{ and } b \in B\}$



Notation: $(a,b) \in R$ or aRb or $R(a,b)$

Example 3. For any function $f : A \to B$, $\mathsf{graph}(f) = \{(x, f(x)) \mid x \in A\}$.

"less than" is a binary relation from $\mathbb{R}$ to $\mathbb{R}$.

Consider the relation teaches $\subseteq$ Instructor $\times$ Courses. It may have tuples of the form
`...(viswanathan, CS173BL1), (viswanathan, CS173AL1), ...(forbes, CS473), (chekuri, CS473) ...`

Cardinality (of finite sets): $|X|$ = number of elements in X.

Example 4. $|\emptyset| =$ $\quad |\{0,1,2,3\}| =$ $\quad |\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}| =$ $\quad |\{0,1,1,2,2\}| =$
$|\{0,1,2\} \times \{a,b,c\}| =$

Proposition 2. *The following statements hold for finite sets A and B.*

1. *If there is a surjective function $f : A \to B$ then $|A| \geq |B|$.*
2. *If there is a injective function $f : A \to B$ then $|A| \leq |B|$.*
3. *If there is a bijective function $f : A \to B$ then $|A| = |B|$.*

Proposition 3. *For a set A such that $|A| = n$, $|\mathsf{pow}(A)| = 2^n$.*