warm-up: simplify \((p \rightarrow q) \land (\neg p \rightarrow q)\)

\[= (\neg p \lor q) \land (p \lor q)\]

\[= (p \land \neg p) \lor q\]

\[= \text{false} \lor q\]

\[= q\]

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Proof by cases:

\((p_1 \lor p_2 \lor \ldots \lor p_k) \land (p_1 \rightarrow q) \land (p_2 \rightarrow q) \land \ldots \land (p_k \rightarrow q)\)

implies

\(q\)

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warm-up: simplify \(\neg p \rightarrow \text{false}\)

\[= p \lor \text{false} = p\]
case 7: Assume \( y \geq 0 \geq x \). Then \( |x| = -x \), \( |y| = y \), \( |xy| = -xy \). Hence, \( |x||y| = -xy = |xy| \).

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Lecture 6: More Proofs
September 9, 2019

**Definition 1.** For a real number \( x \), \( |x| \) is defined as follows.

\[
|x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{otherwise}
\end{cases}
\]

**Problem 1.** For real numbers \( x, y \), \( |xy| = |x||y| \).

We prove this by cases.

**Case 1:** Assume \( x, y > 0 \). Then: \( |x| = x \), \( |y| = y \) and \( |xy| = xy \). Therefore, \( |x||y| = xy = |xy| \).

**Case 2:** Assume \( xy < 0 \). Then \( |x| = -x \), \( |y| = -y \), \( |xy| = xy \). Therefore, \( |x||y| = (-x)(-y) = xy = |xy| \).

**Case 3:** Assume without loss of generality that \( x \geq 0 > y \). Then \( |x| = x \), \( |y| = -y \), \( (xy) = -xy \). Therefore, \( |x||y| = x(-y) = -xy = |xy| \).

**Problem 2.** Prove that \( \sqrt{2} \) is irrational.

We prove this by contradiction. Assume that \( \sqrt{2} \) is rational. Let \( \sqrt{2} = \frac{a}{b} \) be a ratio in simplest form. \( a, b \in \mathbb{Z} \), \( a, b \neq 0 \), and \( a \) and \( b \) have no prime factors in common. Then:

\[
\sqrt{2} = \frac{a}{b} \Rightarrow \sqrt{2} b = a \Rightarrow 2b^2 = a^2 \Rightarrow a^2 \text{ is even}
\]

\( \Rightarrow a \text{ is even} \Rightarrow a = 2k \) for some \( k \in \mathbb{Z} \).

\[
\Rightarrow 2b^2 = (2k)^2 = 4k^2 \Rightarrow b^2 = 2k^2 \Rightarrow b^2 \text{ is even}
\]

\( \Rightarrow b \text{ is even} \)

\( \Rightarrow a \) being even and \( b \) being even contradicts our assumption that \( \frac{a}{b} \) is in simplest form. Hence, \( \sqrt{2} \) is irrational.
Problem 3. There are infinitely many primes.

We prove this by contradiction. Assume that there only finitely many primes. Let $k$ be the number of primes and let $p_1, \ldots, p_k$ the $k$ prime numbers. Consider $m := (p_1 \cdots p_k) + 1$. $m$ is not divisible by $p_i$ for $i \in \{1, \ldots, k\}$. Therefore, $m$ is a prime number greater than $p_1, p_2, \ldots, p_k$. This contradicts our assumption that there are only $k$ primes. Therefore, there must infinitely many primes.

Problem 4. There are irrational numbers $x$ and $y$ such that $x^y$ is rational.

\[ w = \sqrt{2} \]

**Case 1:** $w$ is rational. Then $x = y = \sqrt{2}$ satisfies the requirements.

**Case 2:** $w$ is irrational. Consider $w = \sqrt{2}$

\[ w^2 = (\sqrt{2} \cdot \sqrt{2})^2 = \sqrt{2} \cdot \sqrt{2} = \sqrt{2}^2 = 2 \]

In this case, $x := w$, $y := \sqrt{2}$ satisfies the requirements.
\[ m = \left( \prod_{i=1}^{K} P_i \right) + 1 \]

\[ m = P_1 \left( \prod_{i=2}^{K} P_i \right) + 1 \]

\[ m = P_1 x S + 1 \]

remainder of \( m \) to \( P_1 \) is 1.