~~$\exists x$~~

Lecture 4: Quantifiers and Proofs

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Definition 1. A **predicate** is a proposition that depends on the value of variables.

$P(x)$ = " x is prime" ; $P(2)$ True $P(4)$ False

$Q(n) :=$ " $n^2 - 4 = 0$ " $Q(2)$ True $Q(3)$ F

$R(x,y) :=$ " $x + y = 0$ " $R(1,3)$ F $R(3,-3)$ T

Universal Quantification

$\forall x \in \mathbb{Z}.\ \underline{x^2 \geq 0}$

$\{0, 1, -1, 2, -2, \ldots\}$

For all values of $x \in \mathbb{Z}$, $x^2 \geq 0$

$P(x) = x^2 \geq 0$

$\forall x \in \mathbb{Z}.\ P(x)$

$\equiv P(0)$ AND $P(1)$ AND $P(-1)$ AND $P(2)$ . . . .

Existential Quantification

$\exists x \in \mathbb{Z}.\ \overline{x^2 - 4 = 0} \to T$

$x = 2$

$x = -2$

$\exists x \in \mathbb{R}.\ x^2 - 4 = 0$

$Q(x) \stackrel{\triangle}{=} x^2 - 4 = 0$

$\exists x \in \mathbb{Z}.\ Q(x)$

$Q(0)$ OR $Q(1)$ OR $Q(-1)$ . . . .

Domain Discourse: The set over which variables take values

$\mathbb{Z}$: $\forall x.\ x^2 \geq 0$ $\mathbb{C}$ (complex numbers): $\forall x.\ x^2 \geq 0$

$\mathbb{R}$: $\forall x.\ x^2 \geq 0$

$\forall P \in Prop.\ \forall Q \in Prop.\ \overline{P \text{ AND } Q \equiv Q \text{ AND } P}$

$\forall x \in \mathbb{Z}.\ \exists y \in \mathbb{Z}.\ x + y = 0 \longrightarrow$ T because for any x Take $y = -x$ & $x + y = 0$

~~$\exists y$~~ $\exists y \in \mathbb{Z}\ \forall x \in \mathbb{Z}\ x + y = 0$ X

$\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ x + y = 0$ X

De Morgan's Laws

NOT(P AND Q) = NOT(P) OR NOT(Q)

NOT(P OR Q) = NOT(P) AND NOT(Q)

NOT $(\exists x\ Q(x))$ = $\forall x.$ NOT$(Q(x))$

NOT$(\forall x\ Q(x)) \equiv \exists x.$ NOT$(Q(x))$

NOT$(\exists y \in \mathbb{Z}\ \forall x \in \mathbb{Z}\ x + y = 0) \equiv \forall y \in \mathbb{Z}$ NOT(___)

$\equiv \forall y \in \mathbb{Z}\ \exists x \in \mathbb{Z}\ x + y \neq 0$.

Definition 2. An integer n is **even** if there is an integer k such that $n = 2k$. An integer n is **odd** if there is an integer k such that $n = 2k + 1$.

Problem 1. Prove: If n is an odd integer then n^2 is is odd.

$\forall n \in \mathbb{Z}$. n is odd IMPLIES n^2 is odd.

Assume n is odd.

There is some $k \in \mathbb{Z}$. $n = 2k+1$.

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$$
$$= 2\underbrace{(2k^2 + 2k)}_{\in \mathbb{Z}} + 1$$
$$= 2\ell + 1 \quad \text{where } \ell = 2k^2 + 2k.$$

P	Q	P → Q
F	F	T
F	T	T
T	F	F
T	T	T

$$\text{NOT}(\forall x\ P(x)) = \exists x.\ \text{NOT}(P(x))$$

Definition 3. An integer n is a **perfect square** if there is an integer k such that $n = k^2$. Example: 0, 1, 4, 9, 16, 25, 36 ...

Problem 2. Disprove: Any integer is the sum of two perfect squares.

$\exists n \in \mathbb{Z}$. n is not the sum of two perfect squares.

$n = 3$. Perfect squares < 3: 0, 1 ← Can't add two of these to get 3.

$n = -1$. All perfect squares ≥ 0