Lecture 33: Conditional Probability

Date: November 22, 2019.

Problem 1. When a black and a white die are rolled, the total shown on the two dice is 7. What is the probability that the white die shows 1?

Sample Space $= S = \{ \boxed{(1,1)}, \boxed{(1,2)}, \boxed{(1,3)}, \boxed{(1,4)}, \boxed{(1,5)}, \boxed{(1,6)}$
$(2,1), (2,2), (2,2), (2,4), \boxed{(2,5)}, (2,6)$
$(3,1), (3,2), (3,3), \boxed{(3,4)}, (3,5), (3,6)$
$(4,1), (4,2), \boxed{(4,3)}, (4,4), (4,5), (4,6)$
$(5,1), \boxed{(5,2)}, (5,3), (5,4), (5,5), (5,6)$
$\boxed{(6,1)}, (6,2), (6,3), (6,4), (6,5), (6,6) \}$

$Pr((i,j)) = \frac{1}{36}$

$Pr[(i,j) \mid sum = 7]$

$= \begin{cases} 0 & i+j \neq 7 \\ \dfrac{\frac{1}{36}}{Pr[sum=7]} & i+j = 7 \end{cases}$

$= \frac{1}{36} \div \frac{1}{6} = \frac{6}{36} = \frac{1}{6}$

$Pr[\text{white die} = 1 \mid sum = 7] = \frac{1}{6}$.

Definitions. The probability of an event A given that event B happens is written as $\mathsf{Pr}[A|B]$. It is defined by a new probability space where

- Sample space is the same.
- The new probabiity distribution is given by

$$\mathsf{Pr}[x|B] = \begin{cases} 0 & \text{if } x \notin B \\ \frac{\mathsf{Pr}[x]}{\mathsf{Pr}[B]} & \text{if } x \in B \end{cases}$$

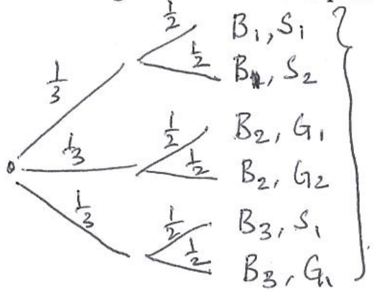
Then $\mathsf{Pr}[A|B] = \sum_{x \in A} \mathsf{Pr}[x|B]$.

Proposition 1. $\boxed{\mathsf{Pr}[A|B] = \frac{\mathsf{Pr}[A \cap B]}{\mathsf{Pr}[B]}}$. $\Rightarrow$ $Pr[A \cap B] = Pr[B] \cdot Pr[A|B]$

$Pr[A|B] = \sum_{x \in A} Pr[x|B] = \sum_{x \in A \cap B} Pr[x|B] + \sum_{x \in A \cap \bar{B}} \cancel{Pr[x|B]}$

$= \sum_{x \in A \cap B} \frac{Pr[x]}{Pr[B]} = \frac{1}{Pr[B]} \sum_{x \in A \cap B} Pr[x] = \frac{Pr[A \cap B]}{Pr[B]}$

Problem 2. There are 3 bags. One bag has 2 silver coins, another has 2 gold coins, and the third has one silver and one gold. One of the 3 bags is chosen at random, and then one of the coins in the bag is picked. The coin turns out to be gold. What is the probability that the other coin in the bag is gold?



Sample Space. $Pr[x] = \frac{1}{6}$

$Gold_1$ = Outcomes where a gold coin picked

$= \{(B_2, G_1), (B_2, G_2), (B_3, G)\}$

$Pr[Gold_1] = \frac{1}{2}$.

$Gold_2$ = Outcomes where 2nd coin is gold

$= \{(B_2, G_1), (B_2, G_2), (B_3, S)\}$

$Pr[Gold_2 \mid Gold_1] = \frac{Pr[Gold_2 \wedge Gold_1]}{Pr[Gold_1]} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$

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Theorem 2 (Bayes Rule). $\Pr[A|B] = \frac{\Pr[A]\Pr[B|A]}{\Pr[B]}$.

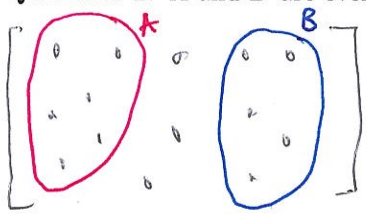
$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\Pr[A] \cdot \Pr[B|A]}{\Pr[B]}.$$

$$\Pr[B|A] = \frac{\Pr[A \cap B]}{\Pr[A]}$$

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B|A]$$

Independent Events. Events A and B are independent if $\Pr[A|B] = \Pr[A]$.

Question 1. A and B are events such that $\Pr[A] \neq 0$ and $A \cap B = \emptyset$. Are A and B independent?



$\Pr[A|B] = 0$ $\qquad$ $\Pr[A] \neq 0$.

Theorem 3. *A is independent of B if and only if* $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$.

($\Rightarrow$) A is independent of B.

$$\Pr[A|B] = \Pr[A].$$

$$\Pr[A \cap B] = \Pr[B] \cdot \Pr[A|B] = \Pr[A] \cdot \Pr[B].$$

($\Leftarrow$) $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$.

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\Pr[A] \cdot \cancel{\Pr[B]}}{\cancel{\Pr[B]}} = \Pr[A].$$

A is independent.