Lecture 32: Probability

Date: November 20, 2019.

Problem 1. Suppose we roll a (fair) black die and a (fair) white die. What is the probability that they sum to 7 or 11?

Outcomes $= \{(i,j) \mid i \in \{1,2,\dots 6\}, j \in \{1,2\dots 6\}\}$

$\{ (1,1), (1,2), (1,3)\ (1,4), (1,5)\ \boxed{(1,6)}$
$(2,1), (2,2) \dots (2,6)$
$\vdots\ \boxed{(5,2)} \qquad \boxed{(5,6)}$
$\boxed{(6,1)} \dots \boxed{(6,5)}\ (6,6) \}$

$\Pr(i,j) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$

$\Pr[\text{either } 7 \text{ or } 11] = \Pr[1,6] + \Pr[2,5] + \Pr[3,4] + \Pr[4,3] + \Pr[5,2] + \Pr[6,1] + \Pr[5,6] + \Pr[6,5]$

$= \frac{8}{36} = \frac{2}{9}.$

Probability Spaces. Consists of

Sample Space, a set S of possible outcomes of an experiment

Probability Distribution, a function $\mathsf{Pr} : S \to [0,1]$ that assigns a positive real weight proportion or probability to each outcome such that $\sum_{x\in S} \mathsf{Pr}[x] = 1$.

An **event** $E \subseteq S$ is a subset of outcomes. The probability of an event E is $\mathsf{Pr}[E] = \sum_{x\in E} \mathsf{Pr}[x]$.

Problem 2. Suppose a biased coin, whose probability of showing heads is q, is tossed 30 times. What is the probabiity of seeing 15 heads?

Sample space $= \{H,T\}^{30} = S \qquad \overbrace{HHTTHT\dots H}^{30}$

$\Pr[\sigma] = q^i(1-q)^{30-i}$ where σ has i coin tosses that result in H.

HHT

$\boxed{\sum_{\sigma\in S} \Pr[\sigma] = 1}$

$E = \{\sigma \in \{H,T\}^{30} \mid \sigma \text{ has } 15 \text{ H's}\}$

$\Pr[\sigma] = q^{15}(1-q)^{15} \quad \forall\, \sigma \in E$

$\Pr[E] = \binom{30}{15} q^{15}(1-q)^{15}$

$\sum_{\sigma\in S}\Pr[\sigma] = \sum_{\sigma: 0H}\Pr[\sigma] + \sum_{\sigma: 1H}\Pr[\sigma] + \dots \sum_{\sigma: 30H}\Pr[\sigma]$

$= \binom{30}{0}(1-q)^{30} + \binom{30}{1} q(1-q)^{29} + \dots + \binom{30}{i} q^i(1-q)^{30-i} + \dots \binom{30}{30} q^{30}$

$= [(1-q)+q]^{30} = 1$

$= \frac{1}{|S|}$

A probability space is said to be **uniform** if $\Pr[x] = \Pr[y]$ for all outcomes x, y. Then $\Pr[E] = \frac{|E|}{|S|}$.

Problem 3. In a class containing 95 students, what is the probability that two people share the same birthday? Assume that all possible birthdays are equally likely.

Birthdays $= 1 \ldots 366$

Sample Space $= S = \{1, \ldots 366\}^{95}$. $|S| = (366)^{95}$

$\Pr[\sigma] = \frac{1}{(366)^{95}}$.

$E = \{\sigma \mid \exists i, j \; i \neq j \; \sigma(i) = \sigma(j)\}$

$\bar{E} = \{\sigma \mid \forall i, j \; i \neq j \Rightarrow \sigma(i) \neq \sigma(j)\}$

$|\bar{E}| = 366 \times 365 \times 364 \times \ldots \times \cancel{271} = \frac{366!}{271!}$

$\Pr[\text{No two share the same birthday}] = \frac{366 \times 365 \times \ldots (366 - n)}{366 \times 366 \times \ldots 366}$

$= \left[\frac{366}{366}\right]\left[\frac{366-1}{366}\right] \ldots \left[\frac{366-n}{366}\right]$

$= 1 \left[1 - \frac{1}{366}\right]\left[1 - \frac{2}{366}\right] \ldots \left[1 - \frac{n}{366}\right]$

$\leq \frac{1}{200{,}000}$

$\Pr[\text{Two people share}] = 1 - \frac{1}{200{,}000} = 0.999999$

Probability Rules from Set Theory.

- **Sum Rule.** If $E_1, E_2, \ldots E_n$ are pairwise disjoint sets, then

$$\Pr[\bigcup_{i=1}^{n} E_i] = \sum_{i=1}^{n} \Pr[E_i]$$

- **Complement Rule.** $\Pr[\overline{A}] = 1 - \Pr[A]$.
- **Difference Rule.** $\Pr[B - A] = \Pr[B] - \Pr[A \cap B]$.
- **Inclusion-Exclusion Rule.** $\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$.
- **Boole's Inequality.** $\Pr[A \cup B] \leq \Pr[A] + \Pr[B]$.
- **Monotonicity Rule.** If $A \subseteq B$ then $\Pr[A] \leq \Pr[B]$.