Lecture 31: Principle of Inclusion-Exclusion and Combinatorial Proofs

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Principle of Inclusion-Exclusion.

- For any sets $A, B$, $|A \cup B| = |A| + |B| - |A \cap B|$
- For any sets $A, B, C$, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- More generally, for any sets $S_1, S_2, \ldots, S_n$, $|S_1 \cup S_2 \cup \ldots \cup S_n| = \sum |S_i| - \sum |S_i \cap S_j| + \sum |S_i \cap S_j \cap S_k| - \ldots + (-1)^{n+1} |S_1 \cap S_2 \cap \ldots \cap S_n| = \sum_{I \subseteq \{1, \ldots, n\}} (-1)^{|I|+1} \left| \bigcap_{i \in I} S_i \right|$

Problem 1. In how many permutations of the set $\{0, 1, 2, \ldots, 9\}$ do either 4 and 2, 0 and 4 or 6 and 0 appear consecutively?

1. $S = \text{set of permutations with the given property}$
2. $A_{42} = \text{set of all permutations where 4 is followed by 2}$
3. $A_{04} = \text{set of all permutations where 0 is followed by 4}$
4. $A_{60} = \text{set of all permutations where 6 is followed by 0}$

$A_{42} \cap A_{04} \cap A_{60}$

- $|A_{42}| = 1$ for permutations of $\{0, 1, 3, 4, 2, 5, 6, 7, 8, 9\}$ is $9!$
- $|A_{04}| = 1$, $A_{60}$
- $|A_{42} \cap A_{04}| = 1$ for permutations of $\{0, 1, 3, 4, 2, 5, 6, 7, 8, 9\}$ is $8!$
- $|A_{42} \cap A_{60}| = 1$ for permutations of $\{0, 1, 3, 4, 2, 5, 6, 7, 8, 9\}$ is $8!$
- $|A_{04} \cap A_{60}| = 1$ for permutations of $\{0, 1, 3, 4, 2, 5, 6, 7, 8, 9\}$ is $7!$
- $|S| = 9! + 9! + 9! - 8! = 9!$
- $|S - (A_{42} \cap A_{04} \cap A_{60})| = 3! - 2! + 1$$

Problem 2. Suppose we have 4 distinct letters to be placed in 4 different pre-addressed envelopes. How many ways can we place letters in envelopes so that no letter is placed in the right envelope?

$S = \text{set of all ways in which the letters can be placed in envelopes}$
$A_i = \text{set of all ways in which letter } i \text{ is placed in correct envelope}$
$A_1A_2A_3A_4 = \text{set of all ways in which at least one letter is placed correctly}$

$|A_1| = 3! = |A_2| = |A_3| = |A_4|$
$|A_1 \cap A_2| = 2! = |A_1 \cap A_3| = |A_2 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_4| = |A_3 \cap A_4|$
$|A_1 \cap A_2 \cap A_3| = 1! = |A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4|$
$|A_1 \cap A_2 \cap A_3 \cap A_4| = 1$
$|A_1A_2A_3A_4| = \binom{4}{2} \cdot 2! + \binom{4}{3} \cdot 1! - \binom{4}{4} \cdot 0!$
$= \frac{4!}{2!} \cdot 3! - \frac{4!}{3!} \cdot 2! + \frac{4!}{3!} \cdot 1! - \frac{4!}{4!} \cdot 0!$
$= 4! \left[ \frac{1}{2} - \frac{1}{3!} + \frac{1}{4!}\right]$
Problem 3. \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

Let \( S \) be a set of \( n \) elements. Let \( A \) be one such element of \( S \).

Define \( B = \) collection of all subsets of \( S \) of size \( k \).

LHS: \( |B| \)

\[ C = \] collection of all subsets of \( S \) of size \( k \) that contain \( A \).

\[ D = \] collection of all subsets of \( S \) of size \( k \) that do not contain \( A \).

\[ |C \cup D| = \emptyset \quad B = \overline{C \cup D} \quad |B| = |C| + |D| \]

\[ |C| = \binom{n-1}{k-1} \quad |D| = \binom{n-1}{k} \]

Problem 4. \( 2^n = \sum_{i=0}^{n} \binom{n}{i} \)

Problem 5. \( \sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1} \)

Let define \( S = \{1, 2, 3, \ldots, n+1\} \)

\( A = \) collection of all subsets of \( S \) of size \( k+1 \)

\[ |A| = \text{RHS}. \]

LHS: \( \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} \).

\( B_i = \) collection of subsets of \( S \) of size \( k+1 \) where the max is \( i \).

\[ |B_i| = \binom{i-1}{k} \] \[ \text{[Any set in } B_i \text{ contains } i \text{ (the max) and } k \text{ other elements from } \{1, 2, \ldots, i-1\}.] \]

\( B_i \cap B_j = \emptyset \) when \( i \neq j \) (as sets in \( B_i \) and \( B_j \) have different maxes).

\[ A = \bigcup_{i=k+1}^{n+1} B_i \quad (k+1 \text{ is the smallest number that can be max of set with } k+1 \text{ elements}) \]

Thus: \( |A| = \text{LHS} = \sum_{i=k+1}^{n+1} |B_i| = \sum_{i=k+1}^{n+1} \binom{i}{k} = \sum_{i=k}^{n} \binom{i}{k} = \text{RHS}. \)