Lecture 31: Principle of Inclusion-Exclusion and Combinatorial Proofs

Date: November 18, 2019.

Principle of Inclusion-Exclusion.

- For any sets A, B, $|A \cup B| = |A| + |B| - |A \cap B|$
- For any sets A, B, C, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.
- More generally, for any sets $S_1, S_2, \dots S_n$,

$|S_1 \cup S_2 \cup \cdots S_n| = |S_1| + |S_2| + \cdots |S_n| - |S_1 \cap S_2| - |S_1 \cap S_3| \cdots + |S_1 \cap S_2 \cap S_3| \cdots - |S_1 \cap S_2 \cap S_3 \cap S_4|$

$$|\bigcup_{i=1}^{n} S_i| = \sum_{\emptyset \neq I \subseteq \{1,\dots n\}} (-1)^{|I|+1} |\bigcap_{i \in I} S_i|$$

Problem 1. In how many permutations of the set $\{0, 1, 2, \dots 9\}$ do either 4 and 2, 0 and 4 or 6 and 0 appear consecutively? $0123456789 \notin S$ $\quad 1234056789 \notin S$ $\quad 1230456789 \in S$

S = set of permutations with the given property.

A_{42} – set of all permutations where 4 is followed by 2

A_{04} – " " " " 0 " " " 4

A_{60} – " " " " 6 " " " 0

$S = A_{42} \cup A_{04} \cup A_{60}$

$1\cancel{2}3\,6042\,789 \in A_{42} \cap A_{04} \cap A_{60}$

$|A_{42}| = |\text{permutations of } \{0,1,3,42,5,6,7,8,9\}| = 9!$

$|A_{04}| = |A_{60}| = 9!$

$|A_{42} \cap A_{04}| = |\text{permutations of } \{1,3,042,5,6,7,8,9\}| = 8! = |A_{04} \cap A_{60}|$

$|A_{42} \cap A_{60}| = |\text{permutation of } \{1,3,42,5,60,7,8,9\}| = 8!$

$|A_{04} \cap A_{42} \cap A_{60}| = |\text{permutations of } \{1,3,6042,5,7,8,9\}| = 7!$

$|S| = 9! + 9! + 9! - 8! - 8! - 8! + 7! = 3 \cdot 9! - 3 \cdot 8! + 7!$

Problem 2. Suppose we have 4 distinct letters to be placed in 4 different pre-addressed envelopes. How many ways can we place letters in envelopes so that no letter is placed in the right envelope?

S – set of all ways in which the letters can be placed in envelopes. $|S| = 4!$

A_i – set of all ways in which letter i is placed in ~~envelope~~ envelope i.

$A_1 \cup A_2 \cup A_3 \cup A_4$ — set of all ways in which at least one letter is placed correctly

$|A_1| = 3! = |A_2| = |A_3| = |A_4|$

$|A_1 \cap A_2| = 2! = |A_1 \cap A_3| = |A_1 \cap A_4| = |A_2 \cap A_3| = |A_2 \cap A_4| = |A_3 \cap A_4|$

$|A_1 \cap A_2 \cap A_3| = 1! = |A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = |A_2 \cap A_3 \cap A_4|$

$|A_1 \cap A_2 \cap A_4| = 1$

$|A_1 \cup A_2 \cup A_3 \cup A_4| = \binom{4}{1} \cdot 3! - \binom{4}{2} \cdot 2! + \binom{4}{3} \cdot 1! - \binom{4}{4} 0!$

$= \frac{4!}{1!\,3!} \cdot 3! - \frac{4!}{2!\,2!} \cdot 2! + \frac{4!}{3!\,1!} \cdot 1! - \frac{4!}{4!\,0!} \cdot 0!$

$= 4! \left[\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}\right]$

$|S| - |A_1 \cup A_2 \cup A_3 \cup A_4| = 4! - 4! \left[\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}\right] = 4! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$

Derangement: $n! \left[1 - \frac{1}{1!} + \frac{1}{2!} \cdots (-1)^n \frac{1}{n!}\right]$

Problem 3. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

S is a set of n elements. Let A to be one such element of S.

B = collection of all subsets of S of size k.

LHS = $|B|$

C = collection of all subsets of S of size k that contain A.

D = collection of all subsets of S of size k that do not contain A.

$C \cap D = \emptyset$. $\quad B = C \cup D \Rightarrow |B| = |C| + |D|$.

$|C| = \binom{n-1}{k-1} \quad |D| = \binom{n-1}{k}$

Problem 4. $2^n = \sum_{i=0}^{n} \binom{n}{i}$

Problem 5. $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$

Let define $S = \{1, 2, 3, 4 \dots n+1\}$.

A = collection of all subsets of size $k+1$

$|A|$ = RHS.

LHS = $\binom{k}{k} + \binom{k+1}{k} + \dots \binom{n}{k}$.

B_i – collection of subsets of S of size $k+1$ where the max is i.

$|B_i| = \binom{i-1}{k}$ [Any set in B_i contains i (the max) and k other elements from $\{1, 2 \dots i-1\}$.]

$B_i \cap B_j = \emptyset$ when $i \neq j$ (as sets in B_i and B_j have different maxes)

$A = \bigcup_{i=k+1}^{n+1} B_i$ ($k+1$ is the smallest number that can be max of set with $k+1$ elements.)

Thus $|A| = \text{LHS} = \sum_{i=k+1}^{n+1} |B_i| = \sum_{i=k+1}^{n+1} \binom{i-1}{k} = \sum_{i=k}^{n} \binom{i}{k} = \text{RHS}$.