Lecture 30: Pigeonhole Principle

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Subset Split Rule/Multinomial Coefficient. The expression
\[
\binom{k_1 + k_2 + \cdots + k_m}{k_1, k_2, \ldots, k_m} = \frac{(k_1 + k_2 + \cdots + k_m)!}{k_1! k_2! \cdots k_m!}
\]
is the number of ways
- of forming \( m \) distinct subsets of sizes \( k_1, k_2, \ldots, k_m \) (respectively) out of a set of \( (k_1 + k_2 + \cdots + k_m) \) elements;
- of the number of sequences formed from \( l_1, l_2, \ldots, l_m \), where the sequence has \( k_1 \) copies of \( l_1 \), \( k_2 \) copies of \( l_2 \), \ldots, \( k_m \) copies of \( l_m \) in the sequence.

Binomial Theorem. \((x + y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.\)

Problem 1. What is the coefficient of \( b^3 k^2 o^0 r^1 \) in the expansion of \((b + e + k + o + p + r)^{10}\)?

\[
\begin{align*}
& (b + e + k + o + p + r)^{10} = \sum_{i_1, i_2, i_3, i_4, i_5, i_6} \binom{10}{i_1, i_2, i_3, i_4, i_5, i_6} b^{i_1} e^{i_2} k^{i_3} o^{i_4} p^{i_5} r^{i_6} \quad \text{Multinomial Theorem} \\
& \binom{a+b}{a} = \binom{a+b}{b}
\end{align*}
\]

Pigeonhole Principle. If \(|A| > |B|\) then for every function \( f : A \to B \), there exist distinct \( a, b \in A \) such that \( f(a) = f(b) \). 

Problem 2. Let \( S \) be any \( n \)-element set of integers. There are \( a, b \in S \) such that

\[
S = \{4, 3, 1, 7, 8, 5\} \quad \text{and} \quad \exists a, b \in S \quad \text{such that} \\
\begin{align*}
& a \neq b \\
& f : S \to \{0, \ldots, n-2\} : f(a) = n \cdot \text{rem}(a, n-1) \\
& |S| = n > 1 \quad \text{and} \quad |0, \ldots, n-2| \\
& \exists a, b, f(a) = n \cdot \text{rem}(a, n-1) = n \cdot \text{rem}(b, n-1) = f(b) \\
& (a, b) \quad a = k(n-1) + r \\
& \quad b = k'(n-1) + r \\
& (n-1) \mid (a-b)
\end{align*}
\]
Problem 3. A chess player trains for a championship by playing practice games over 77 days. She plays at least one game on any day, and plays a total of at most 132 games. Prove that no matter what her schedule of games looks like, there is a period of consecutive days in which she plays exactly 21 games.

\[ a_1 - \#\text{games played on days 1} \ldots i \text{ (inclusive)} \]

\[ 1 \leq a_1 < a_2 < a_3 \ldots < a_{77} \leq 132 \]

\[ 22 \leq a_{1} + 21 < a_{2} + 21 < \ldots < a_{77} + 21 \leq 153 \]

\[ \frac{a_{11}, a_{2}, \ldots, a_{77}, a_{1} + 21, a_{2} + 21, \ldots, a_{77} + 21}{154} \]

By pigeonhole principle \( \exists i, j \) such that \( a_i = a_j + 21 \)

During \( j+1, j+2, \ldots, i \) she plays 21 games.

Generalized Pigeonhole Principle. Let \( B = \{b_1, b_2, \ldots, b_n\} \). Let \( q_1, q_2, \ldots, q_n \in \mathbb{N} \) be such that \( |A| > q_1 + q_2 + \cdots + q_n \). For any function \( f: A \to B \), for some \( i \), \( |\{a \in A | f(a) = b_i\}| > q_i \).

\[ f^{-1}(b) = \{a \in A | f(a) = b\} \]

For \( b_1 \neq b_2 \), \( f^{-1}(b_1) \cap f^{-1}(b_2) = \emptyset \)  \( \implies A = \bigcup_{b \in B} f^{-1}(b) \)

\[ |A| = \sum_{b \in B} |f^{-1}(b_i)| \leq q_1 + \cdots + q_n \]

Contradiction if \( i \implies |f^{-1}(b_i)| \leq q_i \).

- If \( |A| > k|B| \) then for every function \( f: A \to B \), there are \( k + 1 \) distinct elements of \( A a_1, a_2, \ldots, a_{k+1} \) such that for every \( i,j \), \( f(a_i) = f(a_j) \).

Problem 4.

1. How many cards should you pick from a standard deck of 52 cards to guarantee that at least 3 cards of the same suit are chosen? \[ q, \left\lceil \frac{q}{4} \right\rceil \leq 3 \]

2. How many cards should you pick from a standard deck of 52 cards to guarantee that at least 3 cards from the "Hearts" suit are picked? \( A2 \)

Subsequence. For a sequence \( a_1, a_2, \ldots, a_n \), a subsequence is a sequence of the form \( a_{i_1}, a_{i_2}, \ldots, a_{i_k} \) where \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \).

\[ 9, 10, 7, 8, 5, 6, 3, 4, 1, 2 \]

Increasing \[ 9, 10 \]

Decreasing \[ 9, 1, 4, 2 \]

Theorem 1 (Erdös-Szekeres). Any sequence of \( n^2 + 1 \) distinct real numbers contains a subsequence of length at least \( n + 1 \) that is either increasing or decreasing.