Lecture 30: Pigeonhole Principle

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Subset Split Rule/Multinomial Coefficient. The expression

$$\binom{k_1 + \overset{k_2}{\cancel{k+2}} \cdots + k_m}{k_1, k_2, \ldots k_m} = \frac{(k_1 + k_2 + \cdots + k_m)!}{k_1! k_2! \cdots k_m!}$$

is the number of ways

- of forming m distinct subsets of sizes $k_1, k_2, \ldots k_m$ (respectively) out of a set of $(k_1 + k_2 + \cdots + k_m)$ elements;
- of the number of sequences formed from $l_1, l_2, \ldots l_m$, where the sequence has k_1 copies of l_1, k_2 copies of l_2, $\ldots k_m$ copies of l_m in the sequence. BOKEPR Rearrange BOOKKEEPER

Binomial Theorem. $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$.

Problem 1. What is the coefficient of $be^3k^2o^2pr$ in the expansion of $(b+e+k+o+p+r)^{10}$?

$(b+e+k+o+p+r)(b+e+k+o+p+r) \cdots (b+e+k+o+p+r)$

$be^3k^2o^2pr$ $\binom{10}{1,3,2,2,1,1}$

$$(x_1 + x_2 + \cdots x_m)^n = \sum_{i_1+i_2+\cdots i_m = n} \binom{n}{i_1, i_2 \ldots i_m} x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}$$ Multinomial Theorem

$$\binom{a+b}{a, b} = \binom{a+b}{a} = \binom{a+b}{b}$$

Pigeonhole Principle. If $|A| > |B|$ then for every function $f : A \to B$, there exist distinct $a, b \in A$ such that $f(a) = f(b)$. $f^{-1}(b) = \{a \in A \mid f(a) = b\}$; $b_1 \neq b_2$, $f^{-1}(b_1) \cap f^{-1}(b_2) = \emptyset$ $A = \bigcup_{b \in B} f^{-1}(b)$

Problem 2. Let S be any n-element set of integers. There are $a, b \in S$ such that ~~$m \mid (a-b)$~~. $(n-1) \mid (a-b)$

$S = \{4, 3, 1, 7, 8\}$ $\exists a, b \in S.$ $4 \mid (a-b)$

$f : S \to \{0, \ldots n-2\}$: $f(a) = \text{rem}(a, n-1)$

$|S| = n > |\{0, \ldots n-2\}|$

$\exists a, b,$ $f(a) = \text{rem}(a, n-1) = \text{rem}(b, n-1) = f(b)$

$(a \neq b)$ $a = k(n-1) + r$ $b = k'(n-1) + r$

$(n-1) \mid (a-b)$

Problem 3. A chess player trains for a championship by playing practice games over 77 days. She plays at least one game on any day, and plays a total of at most 132 games. Prove that no matter what her schedule of games looks like, there is a period of consecutive days in which she plays **exactly** 21 games.

a_i — # games played on days $1 \ldots i$ (inclusive)

$1 \leq a_1 < a_2 < a_3 < \cdots < a_{77} \leq 132$

$22 \leq a_1 + 21 < a_2 + 21 < \cdots < a_{77} + 21 \leq 153$

$\underbrace{\{a_1, a_2 \ldots a_{77}, a_1+21, a_2+21 \ldots a_{77}+21\}}_{154}$

By pigeon hole principle $\exists i, j \quad a_i = a_j + 21$

During $j+1, j+2 \ldots i$ she plays 21 games.

Generalized Pigeonhole Principle. Let $B = \{b_1, b_2, \ldots b_n\}$. Let $q_1, q_2, \ldots q_n \in \mathbb{N}$ be such that $|A| > q_1 + q_2 + \cdots + q_n$. For any function $f : A \to B$, for some i, $|\{a \in A \mid f(a) = b_i\}| > q_i$.

$f^{-1}(b) = \{a \in A \mid f(a) = b\}$ For $b_1 \neq b_2$, $f^{-1}(b_1) \cap f^{-1}(b_2) = \emptyset$ $A = \bigcup_{b \in B} f^{-1}(b)$

$|A| = \sum_{i=1}^{n} |f^{-1}(b_i)| \leq q_1 + \cdots q_n$ contradiction if $\forall i \ |f^{-1}(b_i)| \leq q_i$

- If $|A| > k|B|$ then for every function $f : A \to B$, there are $k + 1$ distinct elements of A $a_1, a_2, \ldots a_{k+1}$ such that for every i, j, $f(a_i) = f(a_j)$.

Problem 4. 1. How many cards should you pick from a standard deck of 52 cards to guarantee that at least 3 cards of the same suit are chosen? 9. $\left\lceil \frac{9}{4} \right\rceil = 3$

2. How many cards should you pick from a standard deck of 52 cards to guarantee that at least 3 cards from the "Hearts" suit are picked? 42

Subsequence. For a sequence $a_1, a_2, \ldots a_n$, a subsequence is a sequence of the form $a_{i_1}, a_{i_2}, \ldots a_{i_k}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

~~10 9~~ 9, 10, 7, 8, 5, 6, 3, 4, 1, 2

9, 10 increasing

9, 4, 2 decreasing

Theorem 1 (Erdös-Szekeres). *Any sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length at least $n + 1$ that is either increasing or decreasing.*