**Lecture 29: More Counting**

Date: November 13, 2019.

**Permutations.** Number of ways of ordering $r$ objects out of a set containing $n$ objects is

$$P(n, r) = n \times (n-1) \times \cdots \times (n-(r-1)) = \frac{n!}{(n-r)!}.$$  

**Subset Rule.** The number of $k$-element subsets of an $n$-element set is

$${n \choose k} = \frac{n!}{k!(n-k)!}.$$  


- Sequences of length 20
  - Order does not matter $= \frac{20!}{20!} \times \frac{C S C \ldots}{C S C \ldots}$
  - Sequences of 0's & 1's with 20 0's and 41's $= \#$ donut choices $= \begin{pmatrix} 24 \choose 4 \end{pmatrix}$

**Problem 2.** How many non-negative integer solutions does the following equation have?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$${24 \choose 4} \text{ General: Picking } n \text{ elements out of } k \neq n \text{ different types of elements}$$

$${n+k-1 \choose k-1} = \begin{pmatrix} n+k-1 \choose n \end{pmatrix}$$

**Problem 3.** How many outcomes are possible when we roll 5 dice that are differently colored? $6^5$

- $x_i$ = how many dice show i
  - $x_1 + x_2 + \ldots + x_6 = 5$
  - $\begin{pmatrix} 5+6-1 \choose 5 \end{pmatrix} = \begin{pmatrix} 10 \choose 5 \end{pmatrix}$

**Problem 4.** We want to form 4 baseball teams, red, blue, green, and yellow, from 36 players. In how many ways can this be accomplished?

$$\begin{pmatrix} 36 \choose 9 \end{pmatrix} \begin{pmatrix} 27 \choose 9 \end{pmatrix} \begin{pmatrix} 18 \choose 9 \end{pmatrix} \begin{pmatrix} 9 \choose 9 \end{pmatrix} = \frac{36!}{27!9!} \cdot \frac{27!}{18!9!} \cdot \frac{18!}{9!9!} = \frac{36!}{9!9!9!9!}$$

**Number of ways of dividing $n$ players into groups of size $k_1, k_2, k_3, \ldots, k_m$ is**

$$\frac{n!}{k_1! k_2! k_3! \cdots k_m!} = \begin{pmatrix} n \choose k_1, k_2, \ldots, k_m \end{pmatrix}$$
Problem 5. How many ways are there to rearrange the letters of the word BOOKKEEPER?

Choosing teams of sizes 1 (B), 2 (O), 2 (K), 3 (E), 1 (R)

\[
\binom{10}{1,2,2,3,1,1} = \frac{10!}{2!2!3!1!1!}.
\]
Sequences: B0, 02, K1, K2, E1, E2, P1, E3, R = S, |S| = 10!

A = Sequences over BOOKKEEPER

f: S → A is 2!2!3! - 1 - 1

| A | = \frac{10!}{2!2!3!}

Problem 6. What is the coefficient of \(x^k y^{n-k}\) in the expansion of \((x + y)^n\)?

\[(x+y)^n = (x+y)(x+y)(x+y) \cdots (x+y)\]

\[\text{Binomial Theorem: } (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

Problem 7. What is \(\sum_{k=0}^{n} \binom{n}{k}\)?

\[(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\], if \(x = 1, y = 1\)

\[(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} = 2^n\]

Problem 8. What is \(\sum_{i \text{ odd}} \binom{n}{i}\)? What is \(\sum_{i \text{ even}} \binom{n}{i}\)?

\[(-1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i (1)^{n-i}\]

\[\sum_{\text{i odd}} \binom{n}{i} = \frac{1}{2} 2^n = 2^{n-1} = \sum_{\text{i even}} \binom{n}{i} - \sum_{\text{i odd}} \binom{n}{i} = 0\]

Problem 9. What is the coefficient of \(bc^2k^2a^2pr\) in the expansion of \((b + e + k + a + p + r)^n\)?