Lecture 29: More Counting

Date: November 13, 2019.

Permutations. Number of ways of ordering r objects out of a set containing n objects is

$$P(n,r) = n \times (n-1) \times \cdots \times (n-(r-1)) = \frac{n!}{(n-r)!}.$$

Subset Rule. The number of k-element subsets of an n-element set is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}. \qquad \binom{n}{k} = \binom{n}{n-k}$$

Problem 1. How many ways can you pick 20 donuts from a selection of 5 flavors? C, S, J, G, B.

~~P(20, 5)~~ |Sequences of length 20| — 5^{20} $\qquad$ CSC.....| CCS...

Order does not matter $= \frac{5^{20}}{20!}$ ×

|Sequences of 0's & 1's with 20 0's and 4 1's| = # donut choices = $\binom{24}{4}$

Problem 2. How many non-negative integer solutions does the following equation have?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

$\binom{24}{4}$ General: Picking n elements out of k different types of elements

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Problem 3. How many outcomes are possible when we roll 5 dice that are differently colored? 6^5
How many outcomes are possible when we roll 5 identical white dice? $\frac{6^5}{5!}$ ×

x_i — how many dice show i.

$x_1 + x_2 + \ldots x_6 = 5$. $\qquad \binom{5+6-1}{5} = \binom{10}{5}$

Problem 4. We want to form 4 baseball teams, red, blue, green, and yellow, from 36 players. In how many ways can this be accomplished?

$$\binom{36}{9}\binom{27}{9}\binom{18}{9}\cdot\binom{9}{9} = \frac{36!}{\cancel{27!}\,9!}\cdot\frac{\cancel{27!}}{\cancel{18!}\,9!}\cdot\frac{\cancel{18!}}{\cancel{9!}\,9!}\cdot\frac{\cancel{9!}}{9!\,0!} = \frac{36!}{9!9!9!9!}$$

Number of ways of dividing n players into groups of size $k_1, k_2, k_3 \ldots k_m$ is

$$\frac{n!}{k_1!\,k_2!\,k_3!\ldots k_m!} = \binom{n}{k_1, k_2, \ldots k_m}$$

Problem 5. How many ways are there to rearrange the letters of the word BOOKKEEPER?

$\square\ \square\ \ldots\ \square$ Choosing teams of size 1 (B), 2 (O), 2 (K), 3 (E), 1 (P) and 1 (R)

$$\binom{10}{1,2,2,3,1,1} = \frac{10!}{2!\,2!\,3!}$$

Sequences $B\,O_1\,O_2\,K_1\,K_2\,E_1\,E_2\,P\,E_3\,R = S$ $\quad |S| = 10!$

$A =$ Sequences over BOOKKEEPER

$f : S \to A$ is $2!\,2!\,3!$-to-1 $\quad |A| = \dfrac{10!}{2!\,2!\,3!}$

Problem 6. What is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$?

$$(x+y)^n = (x+y)(x+y)(x+y)\ldots(x+y) \qquad \binom{n}{k}$$

<u>Binomial Theorem</u> : $\displaystyle (x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

Problem 7. What is $\sum_{k=0}^{n} \binom{n}{k}$?

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \quad \text{If } x=1,\ y=1$$

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Problem 8. What is $\sum_{i \text{ odd}} \binom{n}{i}$? What is $\sum_{i \text{ even}} \binom{n}{i}$?

$$\cancel{(1-1)^n = \sum_{i=0}^{n} \binom{n}{i} (1)^i (-1)^{n-i}} \qquad (-1+1)^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i (1)^{n-i}$$

$$\sum_{i \text{ even}} \binom{n}{i} = \sum_{i \text{ odd}} \binom{n}{i} = \frac{1}{2} 2^n = 2^{n-1} \qquad = \sum_{i \text{ even}} \binom{n}{i} - \sum_{i \text{ odd}} \binom{n}{i} = 0$$

Problem 9. What is the coefficient of $be^3k^2o^2pr$ in the expansion of $(b+e+k+o+p+r)^{10}$?