# Lecture 28: Permutations and Combinations

Date: November 8, 2019.

**Permutations.** A **permuation/arrangement** of $n$ objects is an ordering of the objects. The number of permutations of $n$ *distinct* objects is $n \times (n-1) \times \cdots \times 1 = n!$.

**Problem 1.** A permutation $(a_1, a_2, a_3, a_4, a_5)$ of $\{1,2,3,4,5\}$ is *heavy-tailed* if $a_1 + a_2 < a_4 + a_5$. How many heavy tailed permutations are there? Examples : $\boxed{1,2,3,4,5}$ $\boxed{2,3,1,4,5}$ Nonex: $\boxed{3,4,5,1,2}$

U – set of all permutations . $|U| = 5!$

H – set of all heavy tailed orderings

L – set of light tailed orders : $a_1 + a_2 > a_4 + a_5$ $\Big]$ $f: H \to L$

$f(a_1, a_2, a_3, a_4, a_5) = a_5, a_4, a_3, a_2, a_1$

E – set of orderings $a_1 + a_2 = a_4 + a_5$. : $|U| = |H| + |L| + |E|$ (since H, L, E are disjoint)

Common sum : 6, 7, 5. $= 2|H| + |E|$

Orderings where common sum 6 : $2 \times 2 \times 2$

$|E| = 3 \times 2 \times 2 \times 2 = 24$. $|H| = \frac{5! - 24}{2} = \frac{120 - 24}{2} = 48$.

**Problem 2.** How many orderings of the top 3 finishers are there, in a 10 horse race?

# ways : $10 \times 9 \times 8$

**Observation.** Number of ways of ordering $r$ objects out of a set containing $n$ objects is

$$P(n,r) = n \times (n-1) \times \cdots \times (n-(r-1)) = \frac{n!}{(n-r)!}.$$

**$k$-to-1 Functions.** A function $f: A \to B$ is $k$-to-1 if exactly $k$ elements of the domain are mapped to every element of the codomain, i.e., for every $b \in B$, $|\{a \in A \mid f(a) = b\}| = k$.

**Division Rule.** If $f: A \to B$ is a $k$-to-1 function then $|A| = k|B|$.

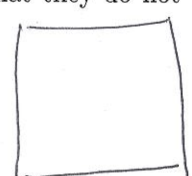
**Problem 3.** How many ways are there to place two identical rooks on a chessboard so that they do not share a row or column?

Positions : $(r_1, c_1, r_2, c_2)$ Positions $\to$ B Config

$(1, 8, 7, 6)$ $\to$ 2-to-1 mapping

$(7, 6, 1, 8)$ $\to$ $|\text{Positions}| = 8 \times 8 \times 7 \times 7$

$|\text{B Config}| = \frac{|\text{Positions}|}{2} = 4 \times 8 \times 7 \times 7 =$



**Problem 4.** How many ways are there to seat $n$ people at a round table?

Fix the position of 1 and order the others $= (n-1)!$

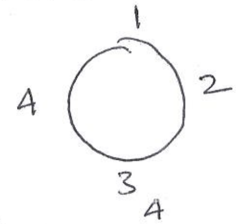
S – set of all orderings of $\{1 \ldots n\}$.

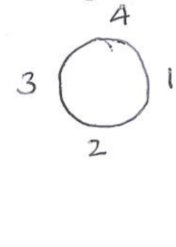
A – set of all seatings in a round table

~~$f(a_1, a_2 \ldots a_n)$~~ $f: S \to A$. Place 1st person & place rest clockwise

$f(a_1, a_2 \ldots a_n) = f(a_n, a_1, a_2 \ldots a_{n-1}) = f(a_{n-1}, a_n, a_1 \ldots a_{n-2})$

$|A| = \frac{|S|}{n} = (n-1)!$



**Problem 5.** Given a standard deck of 52 playing cards, how many hands of 5 cards can one form?

S – Ordering 5 cards from 52 : $P(52,5)$ ] $\text{hand}(a_1 a_2 a_3 a_4 a_5) = \text{hand}(a_2, a_3 a_1, a_4 \ldots)$

A – # 5-card hands.

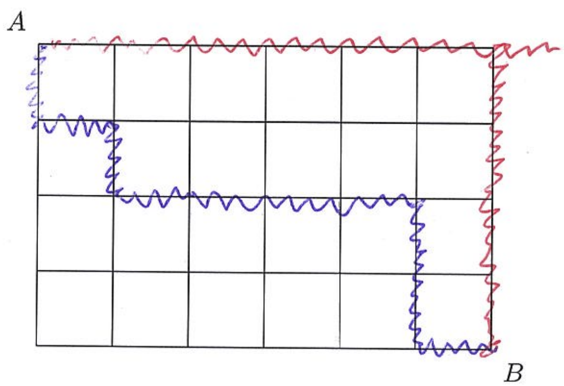
hand : $S \to A$ $5! - 6 - 1$

$|A| = \frac{P(52,5)}{5!} = \frac{52!}{47!\,5!}$

**Subset Rule.** The number of $k$-element subsets of an $n$-element set is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

**Problem 6.** How many shortest routes are there from $A$ to $B$ in the grid-like city plan below?




Path from A to B : Sequence of S and E steps

Red path : ~~SSSSSS~~ EEEEEESSSS

Blue path : SESEEEESSE.

Length of sequence = 10

# S steps = 4.

# paths = $\binom{10}{4}$.

**Problem 7.** How many ways can you pick 20 donuts from a selection of 5 flavors?

**Problem 8.** How many non-negative integer solutions does the following equation have?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 20$$

**Problem 9.** How many outcomes are possible when we roll 5 dice that are differently colored? How many outcomes are possible when we roll 5 identical white dice?