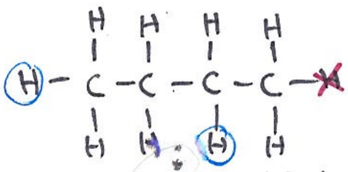
Lecture 20: Trees

Date: October 21, 2019.

Trees

In computer science, there are (at least) two things that people refer to as trees.[1]

A **free (or unrooted) tree** is a connected undirected acyclic graph.



(Butane)

Proposition 1. *Suppose a graph has a closed walk of length > 2. Then the walk contains a cycle.*

Theorem 1. *For any two vertices in a free tree, there is exactly one path connecting them.*

Let u, v be two vertices in a tree T. Because T is a connected graph, there is at least one path connecting u and v. We need to show there is at most one path. Assume there are at least two paths. Follow one of them from u to v and the other one back. This is closed walk. By Proposition 1, there is a cycle.

Proposition 2. *If a free tree has at least two vertices, then it contains at least two vertices with degree one.*

Theorem 2. *A free tree with n vertices has $n - 1$ edges.*

Induction $P(n)$: If a tree T has n vertices, then T has $n-1$ edges.

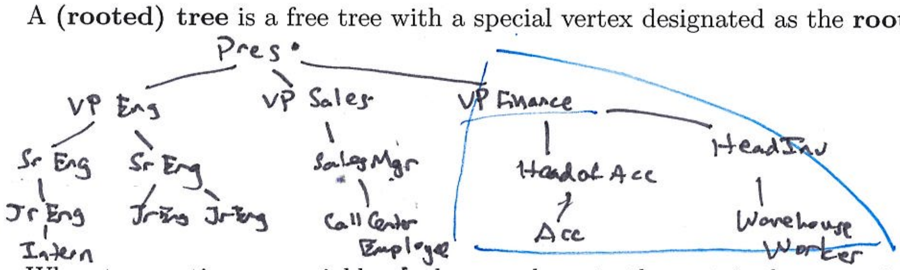
Base Case: $n=1$. A single vertex w/ no edges. # vertices $=1$, # edges $=0$. ✓.

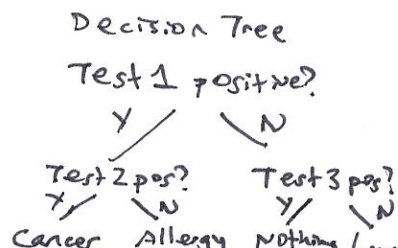
Induction Hypothesis: Assume $P(1), \dots, P(n-1)$, for $n>1$.

Induction Step: Prove $P(n)$. By Proposition 2, I can delete a vertex of degree 1 and its corresponding edge. The result is a tree with $n-1$ vertices. By $P(n-1)$, it has $n-2$ edges. Add back the deleted vertex & edge, we have $n-1$ edges.

(Rooted) Trees

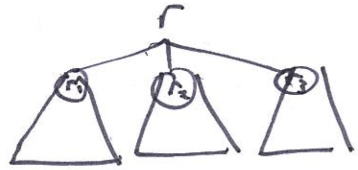
A **(rooted) tree** is a free tree with a special vertex designated as the **root**.



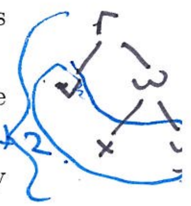


When two vertices are neighbors, the one closer to the root is the **parent**, and the one farther away is the **children**. Children of the same parent are called **siblings**. A vertex with no children is a **leaf**, otherwise it is an **internal node** (or internal vertex). For a vertex v, vertices on the path from v to the root are **ancestors**. Vertices that have v as an ancestor are the **descendants** of v. Given a vertex a in a tree,

[1] Caveat lector: different people have different conventions for what a "tree" with no adjectives refers to.



the **subtree rooted at** a is the tree consisting of a (as the root), all of a's descendants, and all the edges between these vertices.



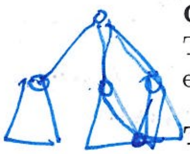
The vertices of a rooted tree can be divided into levels. The **level** of a vertex is the length of the unique path to the root. The **height** of a tree is the maximum level of any leaf.

A rooted tree is an **m-ary tree** if every internal vertex has at most m children. A m-ary tree is **full** if every internal vertex has exactly m children.

$m=2 \Rightarrow$ "binary"

Recursive Definition of and Induction on (Rooted) Trees

Base Case. A single vertex is a (rooted) tree.



Constructor Case. Suppose $T_1, \ldots, T_k$ are rooted trees with roots $r_1, \ldots, r_k$ such that $\bigcap_{i=1}^{k} V(T_i) = \varnothing$. Then the graph formed by taking a root r (that is not a vertex in $T_1, \ldots, T_k$) and adding an edge from r to each of $r_1, \ldots, r_k$ is a tree.

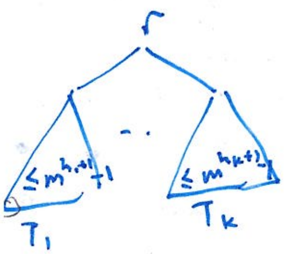
Theorem 3. *For $m \geq 2$, if T is an m-ary tree with height h, then T has at most $m^{h+1} - 1$ vertices.*

$P(T)$: if T is m-ary w/ height h, then $|V(T)| \leq m^{h+1} - 1$.

Base Case: single vertex has height 0.

$1 \leq m^1 - 1$ since $m \geq 2$.

IH: Assume $P(T_1) \cdots P(T_k)$ where $k \leq m$ and $\bigcap_{i=1}^{k} V(T_i) = \varnothing$.



IS: Let T be obtained by making the roots of $T_1 \cdots T_k$ the children of a new root r. Let $h_1 \cdots h_k$ be the heights of $T_1 \cdots T_k$.

Then $|V(T)| = 1 + |V(T_1)| + \cdots + |V(T_k)|$

(IH) $\leq 1 + (m^{h_1+1} - 1) + \cdots + (m^{h_k+1} - 1)$

($h_i \leq h-1$) $\leq k(m^{h-1+1} - 1) + 1$

($k \leq m$) $\leq m^{h+1} + 1 - m \leq m^{h+1} - 1$ (since $m \geq 2$).

Corollary 1. *For $m \geq 2$, if T is an m-ary tree with n vertices, then its height is at least $\log_m(n+1) - 1$.*

Proposition 3. *Consider the family of trees of height at least two whose vertices are colored either orange or blue. If all leaves in a tree are colored blue and the root is colored orange, then there exists an internal node that is colored orange that has a child that is colored blue.*