Lecture ~~20~~ 21: Trees

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Trees

In computer science, there are (at least) two things that people refer to as trees.[1]

A **free (or unrooted) tree** is a connected undirected acyclic graph.

Proposition 1. *Suppose a graph has a closed walk ~~of length ≥ 2~~ consisting of at least two distinct edges. Then the walk contains a cycle.*

Theorem 1. *For any two vertices in a free tree, there is exactly one path connecting them.*

Proposition 2. *If a free tree has at least two vertices, then it contains at least two vertices with degree one.*

Theorem 2. *A free tree with n vertices has $n-1$ edges.*

(Rooted) Trees

A **(rooted) tree** is a free tree with a special vertex designated as the **root**.

When two vertices are neighbors, the one closer to the root is the **parent**, and the one farther away is the **children**. Children of the same parent are called **siblings**. A vertex with no children is a **leaf**, otherwise it is an **internal node** (or internal vertex). For a vertex v, vertices on the path from v to the root are **ancestors**. Vertices that have v as an ancestor are the **descendants** of v. Given a vertex a in a tree,

[1] Caveat lector: different people have different conventions for what a "tree" with no adjectives refers to.

the **subtree rooted at** a is the tree consisting of a (as the root), all of a's descendants, and all the edges between these vertices.

The vertices of a rooted tree can be divided into levels. The **level** of a vertex is the length of the unique path to the root. The **height** of a tree is the level of any leaf.

A rooted tree is an m**-ary tree** if every internal vertex has at most m children. A m-ary tree is **full** if every internal vertex has exactly m children.

Recursive Definition of and Induction on (Rooted) Trees

Base Case. A single vertex is a (rooted) tree.

Constructor Case. Suppose $T_1, \ldots, T_k$ are rooted trees with roots $r_1, \ldots, r_k$ such that $\bigcap_{i=1}^{k} V(T_i) = \varnothing$. Then the graph formed by taking a root r (that is not a vertex in $T_1, \ldots, T_k$) and adding an edge from r to each of $r_1, \ldots, r_k$ is a tree.

Theorem 3. *For $m \geq 2$, if T is an m-ary tree with height h, then T has at most $m^{h+1} - 1$ vertices.*

Corollary 1. *For $m \geq 2$, if T is an m-ary tree with n vertices, then its height is at least $\log_m(n+1) - 1$.*

Proposition 3. *Consider the family of trees of height at least ~~two~~ one whose vertices are colored either orange or blue. If all leaves in a tree are colored blue and the root is colored orange, then there exists an internal node that is colored orange that has a child that is colored blue.*